

Mathematics/ Science/ Technology
An Inter-faculty Second Level Course
Mathematical Models and Methods

mathematical
models and
methods

Unit P
Preparatory Unit

Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

Unit P

Preparatory unit

Prepared by the Course Team

The Open University, Walton Hall, Milton Keynes, MK7 6AA.
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Introduction

MST204 is a course about the application of mathematical methods to the solution of problems which come from other fields: science, technology, economics, and so on. You will learn lots of new mathematics in this course, but the main emphasis is on using mathematics, and not on learning it for its own sake. Mathematical techniques which were topics of interest in their own right in your prerequisite course will now become tools to be used to solve problems. For example, you will already have studied differentiation. In your prerequisite course you probably concentrated on learning how to do it, simply because this would have been the first time you would have met this subject, at least in an Open University course. In *MST204*, on the other hand, differentiation is a tool to be used in the study of how things change: how populations grow and how the planets move around the Sun, to give two examples of many. What is more, the investigation of either of these questions involves not just differentiation but several other mathematical techniques as well, all of which have to be used efficiently in combination.

If you are to use mathematical techniques as tools in this way, you must feel confident of your grasp of them from the outset. This is why *MST204* begins not with new material, but with this unit of preparation. In it you will be able to review the mathematical techniques from prerequisite courses which will be needed for *MST204*. The topics covered in this unit are ones which you should have met before; but it will have been months, if not years, since you last studied many of these things, and though you probably knew them well then, you may not have them quite at your fingertips now.

There are many worked examples and exercises in this unit, and these should form the main focus of your attention. In particular, the more exercises you do, the better. In many places in the unit we have suggested a minimum set of exercises for you to try. If you can manage these without undue difficulty, then by all means move on to the next topic; but if not, try some more from the supplementary set until you do feel confident. But please do try as many as you can (and we mean 'try'—do not just look up the solutions).

The mathematical techniques reviewed in this unit will be required for two purposes in *MST204*. The first is for calculations and for arguments in the unit texts. The second is for exercises and for assessment. The distinction is this: in the first case you have to be able to follow what is written, but in the second you have to be able to carry out the work by yourself. Doing exercises and assessment requires a firmer grasp of the relevant techniques than does following a calculation or a line of argument.

Study guide

The sections may be read in any order. There are two audio-tape activities, in Subsections 1.3 and 4.2. There is no television programme associated with this unit.

Unlike the other units in this course, this one has no unit outline in the *MST204 Handbook*. However, the important results from this unit are all contained in the *Handbook*, in Sections 3 to 7. You may therefore find it helpful to refer occasionally to the *Handbook* while you are working on this unit.

Sections 3 to 7 of the *Handbook* form a compendium of the mathematical results needed in *MST204*. One way of describing this unit is that its purpose is to provide some background to those results and to remind you how they are used. A more ambitious purpose of the unit is to make Sections 3 to 7 of the *Handbook* unnecessary: if you can really master the contents of this unit, you will never need to refer to that part of the *Handbook* again.

1 Algebra

1.1 Algebraic manipulation

We shall begin this preparatory unit with a set of exercises which are designed to remind you of the techniques of algebraic manipulation, with which you may be a little rusty if it is a while since you last used them. A solid foundation in these skills will help you to concentrate on the new ideas introduced in this course.

Example 1

Solve the equation

$$\frac{1+x}{3+x} = \frac{1}{2}$$

for x .

Solution

Cross multiplying gives

$$2(1+x) = 1(3+x).$$

Multiplying out the brackets, we obtain

$$2 + 2x = 3 + x.$$

Collecting together like terms gives

$$2x - x = 3 - 2,$$

i.e. $x = 1$.

Note that alternative methods of solution are possible. For example, we could first put all (non-zero) terms on the left-hand side of the equation to obtain

$$\frac{1+x}{3+x} - \frac{1}{2} = 0.$$

Putting the fractions over a common denominator gives

$$\frac{2(1+x) - 1(3+x)}{2(3+x)} = 0,$$

$$\text{i.e. } \frac{2 + 2x - 3 - x}{2(3+x)} = 0$$

$$\text{or } \frac{x-1}{2(3+x)} = 0.$$

For a fraction to be equal to zero, the numerator must be equal to zero. This leads to the solution $x = 1$, as before. $\square$

Exercise 1

Solve the following equations for x .

$$(i) \quad 3(x+3) - 7(x-1) = 0$$

$$(ii) \quad \frac{2}{1+x} = \frac{3}{2-x}$$

[Solution on page 60]

Example 2

Make the variable y the subject of the equation

$$\frac{y+3}{y+5} = x.$$

Countdown to Mathematics
Subsections 2.4 and 6.4

Countdown to Mathematics
Subsections 2.5 and 6.5

Solution

We have to rearrange the equation so that it takes the form

$$y = \text{expression in } x.$$

We start by cross multiplying to give

$$y + 3 = x(y + 5) = xy + 5x.$$

Collecting together the terms involving y on one side of the equation, and the other terms on the other side, we obtain

$$y - xy = 5x - 3,$$

$$\text{i.e. } (1 - x)y = 5x - 3.$$

$$\text{Hence } y = \frac{5x - 3}{1 - x}.$$

(Note that the equation is not defined for $y = -5$ and for $x = 1$, as these values would involve a division by zero, which is not defined.) $\square$

Exercise 2

- (i) Make the variable t the subject of the equation

$$x = x_0 - \frac{1}{2}gt^2.$$

- (ii) Make the variable x the subject of the equation

$$\sqrt{\frac{x-2}{x+3}} = t.$$

[Solution on page 60]

Supplementary Exercise 1

Make a the subject of the equation

$$s = ut + \frac{1}{2}at^2.$$

[Solution on page 76]

Example 3

Solve the simultaneous equations

$$4x + 3y = -1,$$

$$3x + y = 3.$$

(1)

(2)

Countdown to Mathematics
Subsection 7.5

TM282 Unit 1
Subsection 4.2

Solution

Multiplying Equation (2) by 3 gives

$$9x + 3y = 9.$$

Subtracting Equation (1) from this, we obtain

$$5x = 10,$$

$$\text{i.e. } x = 2.$$

Substituting this value in Equation (2) gives

$$6 + y = 3,$$

$$\text{or } y = -3.$$

So the solution is $x = 2, y = -3$.

Of course, there are other methods of solution. For example, we could have started by subtracting 4 times Equation (2) from 3 times Equation (1). However, whatever method you use, you should obtain the solution $x = 2, y = -3$.

In this type of problem, you should always check your solution by substituting your answer into the original equations. In this case,

$$4x + 3y = 4 \times 2 + 3 \times (-3) = 8 - 9 = -1$$

$$\text{and } 3x + y = 3 \times 2 + (-3) = 6 - 3 = 3,$$

which confirms that $x = 2$, $y = -3$ is the correct answer. $\square$

Exercise 3

Solve the simultaneous equations

$$2x - y = 3,$$

$$3x + y = 2.$$

[Solution on page 60]

Supplementary Exercise 2

Solve the simultaneous equations

$$2u - 5v = 19,$$

$$3u + 4v = -29.$$

[Solution on page 76]

Example 4

Factorize

$$x^2 + x - 6.$$

Solution

As the coefficient of x^2 is 1, we need to find two numbers whose product is -6 and whose sum is 1. The possible decompositions of -6 are $1 \times (-6)$, $(-1) \times 6$, $2 \times (-3)$ and $(-2) \times 3$. So the required numbers are -2 and 3. Hence

$$x^2 + x - 6 = (x - 2)(x + 3).$$

We check our answer by multiplying out the brackets:

$$\begin{aligned} (x - 2)(x + 3) &= x(x + 3) - 2(x + 3) \\ &= x^2 + 3x - 2x - 6 \\ &= x^2 + x - 6. \quad \square \end{aligned}$$

Exercise 4

Factorize the following expressions.

$$(i) \ x^2 - 5x + 4 \quad (ii) \ 4t^2 - 5t - 6 \quad (iii) \ 4z^2 - 25$$

$$(iv) \ x^2 - 2xy - 24y^2 \quad (v) \ \lambda^3 - 3\lambda^2 - 4\lambda \quad (vi) \ u^2 - 6u + 9$$

[Solution on page 60]

Supplementary Exercise 3

Factorize the following expressions.

$$(i) \ x^2 + 6x + 5 \quad (ii) \ 2y^2 + 5y + 3 \quad (iii) \ 3x^2 - x$$

[Solution on page 76]

Example 5

Solve the quadratic equation

$$x^2 + x - 6 = 0.$$

Countdown to Mathematics
Subsections 6.2 and 6.3

TM282 Unit 2
Appendix 1

Countdown to Mathematics
Subsections 6.2 and 6.3

M101 Block I Unit 1
Section 1.1

MS284 Unit 1
Section 1

TM282 Unit 2
Subsection 5.2

Solution

Using the solution to Example 4, we see that the equation is equivalent to

$$(x - 2)(x + 3) = 0.$$

If the product of two numbers is zero, then one of the two numbers must itself be equal to zero. Hence either $x - 2 = 0$ or $x + 3 = 0$. So

$$x = 2 \quad \text{or} \quad x = -3. \quad \square$$

Exercise 5

Use factorization to solve the following quadratic equations.

$$(i) \ x^2 - 5x + 6 = 0 \quad (ii) \ p^2 = 10p - 25 \quad (iii) \ 3y^2 + 8y + 4 = 0$$

[Solution on page 60]

Supplementary Exercise 4

Use factorization to solve the following equations.

$$(i) \ x^2 + 2x = 8 \quad (ii) \ 6\lambda^3 + \lambda^2 - \lambda = 0$$

[Solution on page 76]

Exercise 6

Use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the roots of the quadratic equation

$$ax^2 + bx + c = 0$$

to find the (real number) solutions of the following equations, wherever they exist.

$$(i) \ x^2 - x - 12 = 0 \quad (ii) \ 4y^2 + 4y + 1 = 0 \\ (iii) \ z^2 + z + 2 = 0 \quad (iv) \ \lambda^2 + 5\lambda - 3 = 0$$

[Solution on page 61]

Countdown to Mathematics
Subsection 6.5

The formula for the solution of a quadratic equation arises in the following way. Using

$$(x + k)^2 = x^2 + 2kx + k^2,$$

we can see that the non-constant terms of the quadratic expression $ax^2 + bx + c$ can be written as follows:

$$ax^2 + bx = a \left(x^2 + \frac{b}{a}x \right) = a \left\{ \left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right\} = a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a}.$$

Thus

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} + c.$$

This process is called **completing the square**. Using this form, the quadratic equation

$$ax^2 + bx + c = 0$$

can be written as

$$\left(x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of both sides of this equation, we have

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

$$\text{i.e.} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which is the familiar formula for the roots of a quadratic equation.

M101 Block II Unit 1
Section 1.2

MS284 Unit 6
Subsection 2.2

TM282 Unit 2
Appendix 2

Exercise 7

(i) Express

$$x^2 - 4x - 5$$

in completed-square form.

(ii) Use the completed-square form to find the solutions of the quadratic equation

$$x^2 - 4x - 5 = 0.$$

(iii) Use the completed-square form to find the minimum value of $x^2 - 4x - 5$, and the value of x for which this occurs.

[Solution on page 61]

1.2 The binomial theorem and factorials

You are probably familiar with the expansions of $(a + b)^n$ for low (positive integer) values of n . For example,

$$\begin{aligned} (a + b)^0 &= 1, \\ (a + b)^1 &= a + b, \\ (a + b)^2 &= a^2 + 2ab + b^2, \\ (a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, \\ (a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \\ (a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5, \end{aligned}$$

M101 Block I Unit 2
Section 2.5
MS284 Unit 2
Section 4

where we have included the expansions for $n = 0$ and 1 for later convenience. The coefficients in these expansions can best be found by using **Pascal's triangle**, which is given below.

				1					
			1		1				
		1		2		1			
	1		3		3		1		
	1	4		6		4		1	
	1	5	10		10	5		1	
	1	6	15	20		15	6		1
1	7	21	35	35	21	7		1	
1	8	28	56	70	56	28	8		1

This is constructed by noting that each number is the sum of the two numbers diagonally above it, apart from the 1 at the start and end of each row. We also note that the coefficients in the expansion of $(a + b)^n$ are given by the row whose second term is n .

Exercise 8

Use Pascal's triangle to find the expansions of the following.

(i) $(a + b)^7$ (ii) $(a + b)^8$

[Solution on page 61]

Example 6

Use Pascal's triangle to find the expansion of $(2 - 3x)^4$.

Solution

Here we have $a = 2$, $b = -3x$ and $n = 4$. (Notice that the minus sign in the term $2 - 3x$ means that b must be equal to *minus* $3x$, i.e. we must think of $2 - 3x$ as $2 + (-3x)$.)

Using Pascal's triangle,

$$\begin{aligned} (2 - 3x)^4 &= (2)^4 + 4 \times 2^3 \times (-3x) + 6 \times 2^2 \times (-3x)^2 + 4 \times 2 \times (-3x)^3 + (-3x)^4 \\ &= 16 - 96x + 216x^2 - 216x^3 + 81x^4. \quad \square \end{aligned}$$

Exercise 9

Use Pascal's triangle to find the expansions of the following.

- (i) $(x-1)^5$ (ii) $(1-5x)^4$ (iii) $(3+4x)^3$

[Solution on page 61]

Supplementary Exercise 5

Use Pascal's triangle to find the expansions of the following.

- (i) $(1+x)^4$ (ii) $(2+x)^5$

[Solution on page 76]

The coefficient of the term $a^{n-r}b^r$ in the expansion of $(a+b)^n$ is called a *binomial coefficient* and is written nC_r . The fact that

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \cdots + {}^nC_r a^{n-r}b^r + \cdots + {}^nC_{n-2} a^2b^{n-2} + {}^nC_{n-1} ab^{n-1} + b^n$$

for positive integers n is called the **binomial theorem**. In terms of the binomial coefficients, Pascal's triangle is as follows.

$$\begin{array}{ccccccc} & & & & {}^0C_0 & & \\ & & & {}^1C_0 & & {}^1C_1 & \\ & & {}^2C_0 & & {}^2C_1 & & {}^2C_2 \\ & {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 \\ {}^4C_0 & & {}^4C_1 & & {}^4C_2 & & {}^4C_3 & & {}^4C_4 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

So the binomial coefficient nC_r is the r th entry in the n th row of Pascal's triangle, where in each case we start counting with 0. A binomial coefficient can be expressed as a formula by using the **factorial** notation. For the positive integer n , we define n factorial, written $n!$, by

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

In addition, we also define

$$0! = 1.$$

Using this notation,

$${}^nC_r = \frac{n!}{r!(n-r)!}.$$

You may find that your calculator has a key $\boxed{x!}$ for factorials.

Exercise 10

Evaluate the following.

- (i) $5!$
 (ii) $7!$
 (iii) 6C_3 (Check your answer by using Pascal's triangle.)
 (iv) ${}^{20}C_2$

Exercise 11

- (i) Show that

$${}^nC_r = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+2) \times (n-r+1)}{r \times (r-1) \times (r-2) \times \cdots \times 3 \times 2 \times 1}.$$

- (ii) Hence find expressions for nC_1 and nC_2 .
 (iii) Use the formula in part (i) to evaluate ${}^{20}C_3$.

Note that there are r terms on both top and bottom of this quotient.

[Solutions on page 61]

1.3 Powers and logarithms (Audio-tape Subsection)

Exercise 12

Describe the behaviour of a^n for increasing positive integer values of n for the following.

- (i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$ (iv) $a = 0$
(v) $-1 < a < 0$ (vi) $a = -1$ (vii) $a < -1$

In particular, state in each case what happens to a^n as n increases indefinitely.

[Hint: You may find it useful to plot the values of a^n for $n = 1, 2, 3, 4, 5$ for typical values of a .]

[Solution on page 61]

The tape associated with this subsection considers the properties of the **power** function $y = a^x$, and its inverse, the **logarithm** function $y = \log_a x$. You will need some paper, a pen or pencil, the *MST204 Handbook* and a pocket calculator as you listen to the tape. Note that the detailed solutions to the exercises in the audio-tape frames are given directly after the frames.

MS284 Unit 2
Section 1

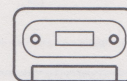
Countdown to Mathematics
Subsections 9.1 to 9.4

M101 Block II Unit 2
Sections 2.3 to 2.5

MS284 Unit 2
Sections 2 and 3

TM282 Unit 6
Sections 4 and 7

Start the audio-tape when you are ready.



1 Integer Powers

base $\nearrow$ $a^n = \underbrace{a \times a \times \dots \times a}_{n \text{ times}}$ $\nwarrow$ power or exponent

$n > 0$

$$a^{-n} = \frac{1}{a^n} = \frac{1}{\underbrace{a \times a \times \dots \times a}_{n \text{ times}}}$$

$$\begin{aligned} a^1 &= a \\ a^2 &= a \times a \\ a^3 &= a \times a \times a \\ &\vdots \end{aligned}$$

$$\begin{aligned} a^{-1} &= 1/a \\ a^{-2} &= 1/a^2 \\ &\vdots \end{aligned}$$

NB $(-a)^n$ is not the same as $-a^n$;
e.g. $(-3)^2 = 9 \neq -9 = -3^2$.

2 Floating Point Notation

$a = 10$

$$23\,500 = 2.35 \times 10^4$$

$$0.0235 = 2.35 \times 10^{-2}$$

Number has the form

$$\pm b \times 10^c$$

c is an integer
 $1 \leq b < 10$

Exercises

Put the following numbers into floating point form.

(i) 54.2

(ii) 0.007 68

(iii) -279.3

(iv) -0.04

3 Power Functions

$$y = a^x$$

any real number

base a must be positive

Important properties

• Multiplication

$$a^p \times a^q = a^{p+q}$$

$$10^2 \times 10^3 = 10^{2+3} \\ = 10^5$$

• Division

$$a^p \div a^q = a^{p-q}$$

$$10^5 \div 10^3 = 10^{5-3} \\ = 10^2$$

• Power of power

$$(a^p)^q = a^{pq}$$

$$(10^2)^3 = 10^{2 \times 3} \\ = 10^6$$

4 Further Properties

(i) Zero power

$$a^0 = 1$$

$$17^0 = 1$$

(ii) Power of 1

$$1^x = 1$$

$$1^{100} = 1$$

(iii) Root

$$a^{1/n} = \sqrt[n]{a}$$

$$27^{1/3} = \sqrt[3]{27} = 3$$

n integer
n > 0

(iv) Rational power

$$a^{m/n} = \begin{cases} (a^m)^{1/n} = \sqrt[n]{a^m} \\ (a^{1/n})^m = (\sqrt[n]{a})^m \end{cases}$$

$$4^{3/2} = \left\{ \begin{array}{l} (4^3)^{1/2} = \sqrt{64} \\ (4^{1/2})^3 = 2^3 \end{array} \right\} = 8$$

m, n positive integers

(v) Negative power

$$a^{-x} = \frac{1}{a^x}$$

$$4^{-3/2} = \frac{1}{4^{3/2}} = \frac{1}{8}$$

(vi) Product

$$(ab)^x = a^x b^x$$

$$(5 \times 3)^7 = 5^7 \times 3^7$$

(vii) Quotient

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$\left(\frac{5}{3}\right)^7 = \frac{5^7}{3^7}$$

5 Exercises

Simplify the following.

(i) $x^2 x^5$

(ii) x^3/x^4

(iii) $(x^2)^3$

(iv) $9^{1/2}$

(v) 2^{-1}

(vi) $8^{-1/3}$

(vii) $16^{3/4}$

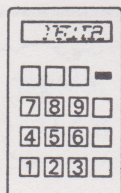
(viii) $\left(\frac{4}{9}\right)^{1/2}$

(ix) $(8x^3)^{1/3}$

6 Exponential Function

$$y = e^x = \exp(x)$$

$$e = 2.718\,28\dots$$



Evaluate:

• e^0

• $e^1 = e$

• e^2

• e^{-2}

• $e^{1/3}$

Sketch the graphs.

$$y = e^x$$

8

1

-2 -1 0 1 2 x

$$y = e^{-x}$$

8

1

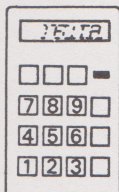
-2 -1 0 1 2 x

Check this
from Section 5.2
of the Handbook

7 Logarithm Function

$$y = \log_e x = \ln x$$

Natural $\log_e$: inverse
of $y = e^x$



Evaluate:

$\bullet \log_e 1$

$\bullet \log_e 2$

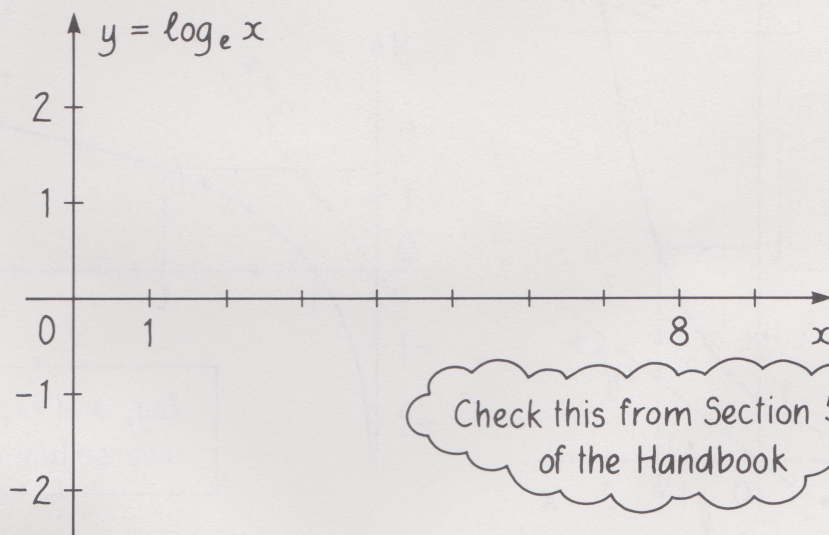
$\bullet \log_e \left(\frac{1}{2}\right)$

$\bullet \log_e (-2)$

Similarly,

$y = \log_a x$
is the inverse of
 $y = a^x$ ($a > 1$).

Sketch
the graph.



Check this from Section 5.2
of the Handbook

8 Properties

These apply for logs to any base.

(i) Addition

$$\log_a u + \log_a v = \log_a (uv)$$

$$\log_a 2 + \log_a 3 = \log_a 6$$

(ii) Subtraction

$$\log_a u - \log_a v = \log_a \left(\frac{u}{v}\right)$$

$$\log_a 12 - \log_a 4 = \log_a 3$$

$$u > 0$$

$$v > 0$$

$$x > 0$$

(iii) Multiple

$$n \log_a x = \log_a (x^n)$$

$$2 \log_a 3 = \log_a 9$$

(iv) Negative

$$-\log_a x = \log_a \left(\frac{1}{x}\right)$$

$$-\log_a 4 = \log_a \left(\frac{1}{4}\right)$$

(v) Log of 1

$$\log_a 1 = 0$$

9 Exercises

Simplify the following.

(i) $\log_e 7 + \log_e 5$

(ii) $\log_e 8 - \log_e 4$

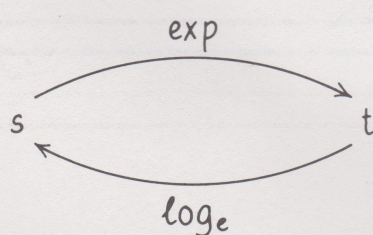
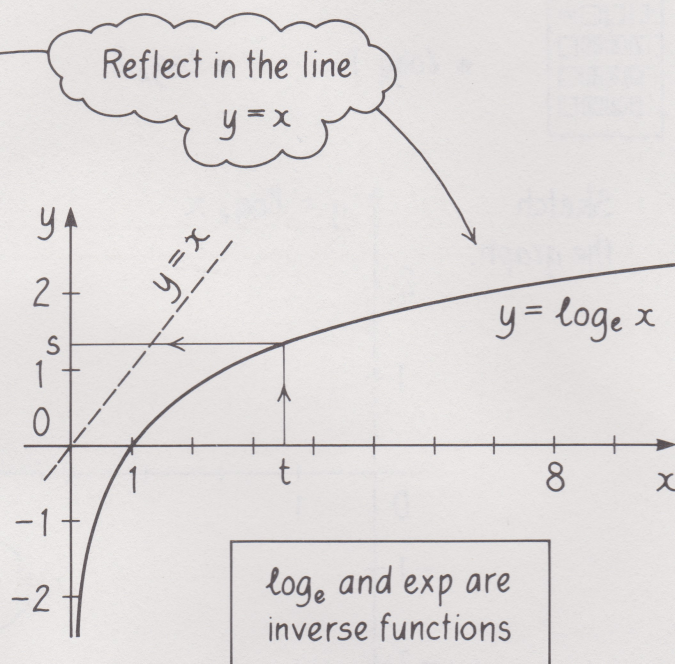
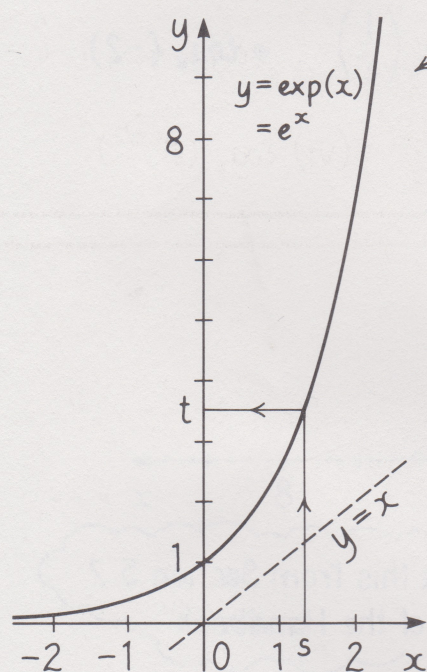
(iii) $3 \log_e 2$

(iv) $-\log_e 5$

(v) $3 \log_e 4 - \log_e 8$

(vi) $\frac{1}{2} \log_e 25 + 3 \log_e \frac{1}{2}$

10 $\log_e$ and Exp



$$t = \exp s \longleftrightarrow s = \log_e t$$

$$t = \exp(\log_e t)$$

$$s = \log_e(\exp s)$$

$$\exp(\log_e x) = \log_e(\exp x) = x$$

with $t = f(x)$: $e^{\log_e f(x)} = f(x)$

Special case
with $x = 1$:
 $e^0 = \log_e e = 1$

11 Using the Inverse PropertiesExamples

(i) $e^{\log_e(x^3)} = x^3$ (ii) $e^{2\log_e x} = e^{\log_e(x^2)} = x^2$

(iii) $e^{[2\log_e x - \log_e(x^2+1)]} = e^{\log_e[x^2/(x^2+1)]} = \frac{x^2}{x^2+1}$ (iv) $\log_e(e^{3x}) = 3x$

Exercises

Simplify the following.

(i) $e^{3\log_e x}$

(ii) $e^{-3\log_e x}$

(iii) $e^{[\log_e x + \log_e(x+1)]}$

(iv) $e^{[\log_e x - 2\log_e(x+1)]}$

(v) $e^{[4+2\log_e(x+3)]}$

(vi) $\log_e(e^{-x/2})$

12 Using Logs to Solve an Equation

$a^x = b$

a and b given; find x

Take logs of both sides:

$$x \log_e a = \log_e b \longrightarrow x = \frac{\log_e b}{\log_e a}$$

ExerciseSolve the equation $1.1^x = 2$.**13** Why e?Derivatives of exp and $\log_e$:

$\frac{d}{dx}(e^x) = e^x;$

$\frac{d}{dx}(\log_e x) = \frac{1}{x}.$

Integrals:

$\int e^x dx = e^x + c;$

$\int \frac{1}{x} dx = \log_e x + c.$

x > 0

Solutions to Exercises in the Tape Frames

Frame 2

- (i) 5.42×10^1 (ii) 7.68×10^{-3} (iii) -2.793×10^2 (iv) -4×10^{-2}

The properties of the power and logarithm functions are summarized in Subsection 3.4 of the *Handbook*.

Frame 5

- (i) $x^2 x^5 = x^{2+5} = x^7$
 (ii) $x^3/x^4 = x^{3-4} = x^{-1} = 1/x$
 (iii) $(x^2)^3 = x^{2 \times 3} = x^6$
 (iv) $9^{1/2} = \sqrt{9} = 3$
 (v) $2^{-1} = \frac{1}{2}$
 (vi) $8^{-1/3} = 1/(8^{1/3}) = 1/\sqrt[3]{8} = \frac{1}{2}$
 (vii) $16^{3/4} = (16^{1/4})^3 = (\sqrt[4]{16})^3 = 2^3 = 8$

(If you start by saying $16^{3/4} = (16^3)^{1/4}$, you will obtain the same answer but the numbers are much larger in the intermediate working.)

- (viii) $(\frac{4}{9})^{1/2} = \frac{4^{1/2}}{9^{1/2}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$
 (ix) $(8x^3)^{1/3} = \sqrt[3]{8x^3} = \sqrt[3]{8} \times \sqrt[3]{x^3} = 2x$

Frame 9

- (i) $\log_e 7 + \log_e 5 = \log_e (7 \times 5) = \log_e 35$
 (ii) $\log_e 8 - \log_e 4 = \log_e (\frac{8}{4}) = \log_e 2$
 (iii) $3 \log_e 2 = \log_e (2^3) = \log_e 8$
 (iv) $-\log_e 5 = \log_e (5^{-1}) = \log_e \frac{1}{5}$
 (v) $3 \log_e 4 - \log_e 8 = \log_e (4^3) - \log_e 8 = \log_e 64 - \log_e 8 = \log_e (\frac{64}{8}) = \log_e 8$
 (vi) $\frac{1}{2} \log_e 25 + 3 \log_e \frac{1}{2} = \log_e (25)^{1/2} + \log_e (\frac{1}{2})^3$
 $= \log_e 5 + \log_e (\frac{1}{8}) = \log_e (5 \times \frac{1}{8}) = \log_e \frac{5}{8}$

Frame 11

- (i) $e^{3 \log_e x} = e^{\log_e (x^3)} = x^3$
 (ii) $e^{-3 \log_e x} = e^{-\log_e (x^3)} = e^{\log_e (1/x^3)} = \frac{1}{x^3}$
 (iii) *Either* $e^{[\log_e x + \log_e (x+1)]} = e^{\log_e [x(x+1)]} = x(x+1)$
or $e^{[\log_e x + \log_e (x+1)]} = e^{\log_e x} e^{\log_e (x+1)} = x(x+1).$
 (iv) *Either* $e^{[\log_e x - 2 \log_e (x+1)]} = e^{[\log_e x - \log_e \{(x+1)^2\}]} = e^{\log_e [x/(x+1)^2]} = \frac{x}{(x+1)^2}$
or $e^{[\log_e x - 2 \log_e (x+1)]} = e^{\log_e x} / e^{2 \log_e (x+1)} = e^{\log_e x} / e^{\log_e \{(x+1)^2\}} = \frac{x}{(x+1)^2}.$
 (v) $e^{[4 + 2 \log_e (x+3)]} = e^4 e^{2 \log_e (x+3)} = e^4 e^{\log_e \{(x+3)^2\}} = e^4 (x+3)^2$
 (vi) $\log_e (e^{-x/2}) = -\frac{x}{2}$

Frame 12

In order to solve the equation

$$1.1^x = 2,$$

we take the natural logarithm of both sides to obtain

$$\log_e (1.1^x) = \log_e 2$$

or

$x \log_e 1.1 = \log_e 2.$

Hence

$x = \frac{\log_e 2}{\log_e 1.1} \simeq 7.2725.$

End of solutions to tape frame exercises.

Supplementary Exercise 6

Write the following in floating-point form.

- (i) 0.001 762 51 (ii) -275

Supplementary Exercise 7

Which is the greater, $(-5)^4$ or $(-4)^5$?

Supplementary Exercise 8

Simplify the following.

- (i) $\left(\frac{16}{x^{12}}\right)^{-1/4}$ (ii) $\frac{1}{3} \log_e 27 + \frac{1}{2} \log_e \frac{1}{9}$ (iii) $\exp \left\{ \frac{5}{2} \log_e (x^4) \right\}$

[Solutions on page 76]

1.4 The summation notation and series

The **summation symbol**, $\sum$, is a convenient shorthand for representing the sum of a number of terms. Specifically, if $a_1, a_2, \dots, a_n$ is a sequence of numbers, then

$\sum_{r=1}^n a_r$ means $a_1 + a_2 + \dots + a_{n-1} + a_n$

and $\sum_{r=m}^n a_r$ means $a_m + a_{m+1} + \dots + a_{n-1} + a_n,$

where m is an integer less than n . So, for example, we can write the sum

$1 + 2^2 + 3^2 + \dots + 99^2 + 100^2$ conveniently as $\sum_{r=1}^{100} r^2$, as $1^2 = 1$, and the infinite sum

$1 + x + x^2 + \dots$ as $\sum_{r=0}^{\infty} x^r$, as $x^0 = 1$. The symbol r is called a *dummy symbol*: we could equally well use any other symbol, as long as it is not used elsewhere in the expression.

Σ is the Greek capital letter sigma.

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MS284 Unit 7
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Exercise 13

Express the following sums using the summation symbol.

(i) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

(ii) $1 + 2^3 + 3^3 + \dots + (N - 1)^3 + N^3$

(iii) $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$

(iv) $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(v) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

[Hint: $(-1)^r = \begin{cases} -1, & r \text{ odd,} \\ +1, & r \text{ even.} \end{cases}$]

[Solution on page 62]

In some cases we can find a simple formula for sums. For example,

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1),$$
$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$
and
$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2.$$

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MS284 Unit 11
Section 7
TM287 Unit 9
Subsection 2.1

Supplementary Exercise 9

Express the sum

$$5^2 + 10^2 + 15^2 + \cdots + 95^2 + 100^2$$

using the summation symbol, and hence evaluate the sum.

[Solution on page 76]

Another sum we can evaluate is the **geometric series**

$$\sum_{r=1}^n ax^{r-1} = a + ax + ax^2 + \cdots + ax^{n-2} + ax^{n-1} = \frac{a(1-x^n)}{1-x} \tag{3}$$

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Subsections 2.2 and 2.3

(where $x \neq 1$). Note that each term in the sum is derived from the previous one by multiplying by x , which is called the **common ratio**. The first term is a and the number of terms is n .

Exercise 14

The initiator of a chain letter sends out five letters to different people. Each of these recipients sends out five letters in turn, and so on. Calculate the total number of people in the chain after ten sets of postings, assuming that no person is in the chain more than once.

[Solution on page 62]

Due to the term x^n in Equation (3), the sum of a geometric series becomes increasingly large in magnitude if $|x| > 1$. However, if $|x| < 1$, then $\lim_{n \rightarrow \infty} x^n = 0$, which means that in this case the infinite geometric series has a finite sum, namely

$$\sum_{r=1}^{\infty} ax^{r-1} = a + ax + ax^2 + \cdots = \frac{a}{1-x} \quad (-1 < x < 1).$$

In particular,

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad (-1 < x < 1).$$

Exercise 15

Evaluate the sum of the infinite geometric series

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots.$$

Exercise 16

By writing the recurring decimal 0.1111... as

$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots,$$

find its value as a fraction.

[Solutions on page 63]

Supplementary Exercise 10

Find the value of the recurring decimal 0.9999... as a fraction.

[Solution on page 76]

2 Trigonometry

2.1 The trigonometric functions

In this course we shall almost always express angles in **radians**. Recall that 360 degrees is 2π radians, i.e. 1 radian is $360/2\pi$ degrees.

Exercise 1

- (i) Express the following angles in radians.
(a) 30° (b) 90° (c) 135° (d) 240° (e) 360°
- (ii) Express the following angles (given in radians) in degrees.
(a) $\frac{1}{4}\pi$ (b) $\frac{1}{3}\pi$ (c) $\frac{2}{3}\pi$ (d) π (e) $\frac{3}{2}\pi$

[Solution on page 63]

You will previously have met the three **trigonometric functions** sine, cosine and tangent. There are three further trigonometric functions, called secant (abbreviated as sec), cosecant (abbreviated as cosec) and cotangent (abbreviated as cot), which are defined by

$$\sec \theta = \frac{1}{\cos \theta}, \quad \operatorname{cosec} \theta = \frac{1}{\sin \theta}, \quad \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}.$$

Although these definitions are quoted in Subsection 4.1 of the *MST204 Handbook*, they are worth remembering, particularly as most calculators do not have keys for sec, cosec and cot and so values have to be found by using their relationships to cos, sin and tan.

It is also worthwhile remembering the sine, cosine and tangent of $0, \frac{1}{6}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi$ and $\frac{1}{2}\pi$ radians. These are given in Table 1, although you may find the triangles in Figures 1 and 2 useful in helping you to remember them.

Table 1

θ (in radians)	θ (in degrees)	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0°	0	1	0
$\frac{1}{6}\pi$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{1}{4}\pi$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{1}{3}\pi$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{1}{2}\pi$	90°	1	0	not defined

The graphs of the sine, cosine and tangent functions are shown in Figures 3 to 5.

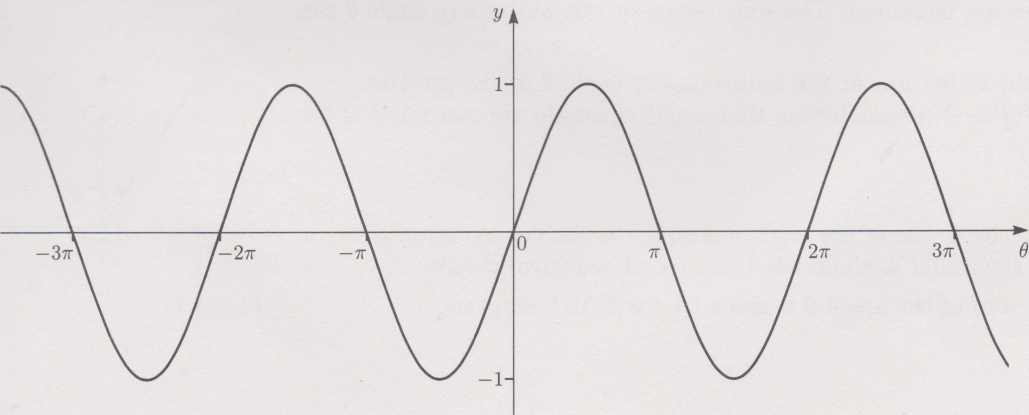


Figure 3 The sine function.

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Subsection 2.1

TM282 Unit 3
Subsection 2.6

Countdown to Mathematics
Subsections 8.1 and 8.2

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Section 3.2

MS284 Unit 3
Section 2

TM282 Unit 3
Subsections 3.2 and 3.3

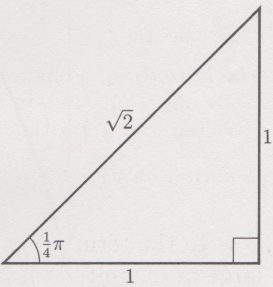


Figure 1 A 45° - 45° - 90° triangle.

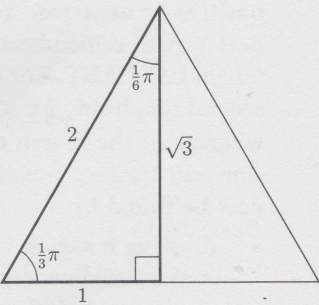


Figure 2 A 30° - 60° - 90° triangle.

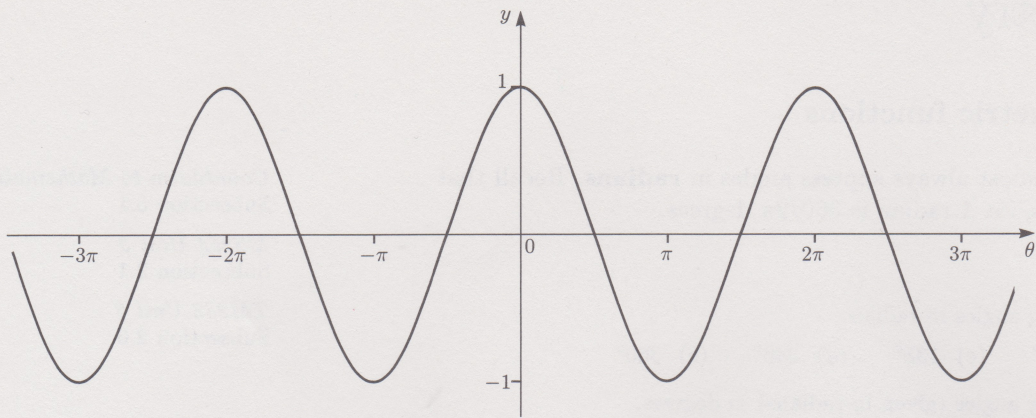


Figure 4 The cosine function.

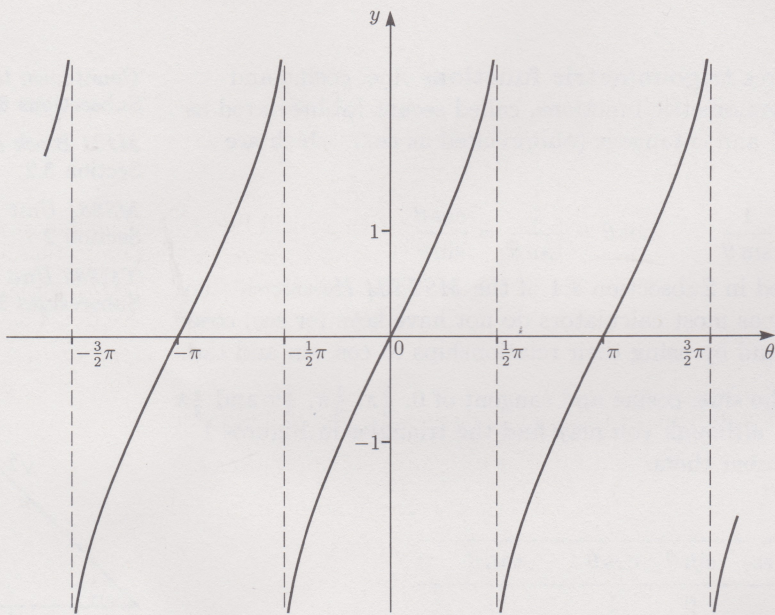


Figure 5 The tangent function.

It is also useful to remember the range of values of θ for which $\sin \theta$, $\cos \theta$ and $\tan \theta$ are positive or negative. You can do this by referring to Figures 3 to 5. But probably the best way of remembering them is by using Figure 6, which you can recall by using the mnemonic CAST. For example, it indicates that the sine function is positive in the second quadrant, $\frac{1}{2}\pi < \theta < \pi$ (and the cosine and tangent functions are negative), whereas in the fourth quadrant, $\frac{3}{2}\pi < \theta < 2\pi$, the cosine function is positive (and the sine and tangent functions are negative). The sine, cosine or tangent of any angle θ can now be found by

- drawing a sketch of the radial line at the anticlockwise angle θ to the positive x -axis, as shown in Figure 7 (remembering that negative angles are measured in the clockwise direction).

Then

- the magnitude of $\sin$, $\cos$ or $\tan$ of the angle θ is equal to $\sin$, $\cos$ or $\tan$ of the acute angle between the radial line and the (positive, or negative) x -axis;
- the sign of $\sin$, $\cos$ or $\tan$ of the angle θ is given by the CAST diagram.

S(in)	A(ll)
T(an)	C(os)

Figure 6 The CAST diagram.

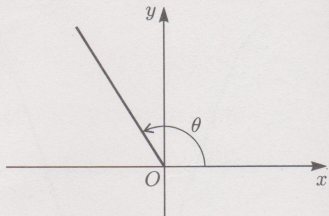


Figure 7

Exercise 2

Write down the values of the following.

- (i) $\sec \frac{1}{6}\pi$ (ii) $\operatorname{cosec}(-\frac{1}{4}\pi)$ (iii) $\cos \frac{5}{4}\pi$ (iv) $\tan(-\frac{11}{6}\pi)$

[Note: Answers of the form $1/\sqrt{2}$, for example, are acceptable.]

[Solution on page 63]

Supplementary Exercise 1

Write down the values of the following.

- (i) $\cot \frac{1}{3}\pi$ (ii) $\sin(-\frac{2}{3}\pi)$

[Solution on page 76]

2.2 The inverse trigonometric functions

Exercise 3

Find all the angles θ in the range $0 \leq \theta < 2\pi$ for which $\sin \theta = \frac{1}{2}$.

[Solution on page 63]

If $x = \sin \theta$ then θ can be described as ‘an angle whose sine is x ’. But there will be more than one angle θ whose sine is x . For example, as you saw in the above exercise, if $\sin \theta = \frac{1}{2}$, then θ can be $\frac{1}{6}\pi$ or $\frac{5}{6}\pi$, as well as either of these angles plus any integer multiple of 2π . To overcome this confusion we define the inverse trigonometric function $\arcsin x$ to be the angle θ in the range $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$ whose sine is x . This is called the principal value range of the function $\arcsin x$. The notation $\sin^{-1} x$ is sometimes used for $\arcsin x$, but we shall not use this notation in this course to avoid possible confusion with $\frac{1}{\sin x}$. The trigonometric functions cosine and tangent also have corresponding inverse functions. They are all listed below, together with their corresponding principal value ranges.

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Subsection 3.2

Table 2

Function	Inverse function	Principal value range of inverse function
$x = \sin \theta$	$\theta = \arcsin x$	$-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$
$x = \cos \theta$	$\theta = \arccos x$	$0 \leq \theta \leq \pi$
$x = \tan \theta$	$\theta = \arctan x$	$-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$

Exercise 4

Find the values of the following.

- (i) $\arccos \frac{1}{2}$ (ii) $\arctan 1$ (iii) $\arccos(-\frac{1}{2})$ (iv) $\arctan(-1)$

[Solution on page 63]

Supplementary Exercise 2

Evaluate the following.

- (i) $\arcsin(\sin \frac{5}{3}\pi)$ (ii) $\sin(\arccos \frac{1}{2})$

[Solution on page 76]

2.3 Some useful trigonometric identities

You will previously have met the trigonometric identity

$$\sin^2 \theta + \cos^2 \theta = 1.$$

(1)

This is equivalent to Pythagoras' theorem for acute angles θ , as shown in Figure 8. Dividing both sides of Equation (1) by $\cos^2 \theta$, we obtain

$$\frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \frac{1}{\cos^2 \theta},$$

i.e. $1 + \tan^2 \theta = \sec^2 \theta.$

Similarly, by dividing by $\sin^2 \theta$, we can obtain a third identity,

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta.$$

Given the value of one of $\sin \theta$, $\cos \theta$ and $\tan \theta$, we can find the possible values of the other two by using these identities.

Example 1

If θ is the acute angle such that $\tan \theta = \frac{3}{4}$, find the values of $\cos \theta$ and $\sin \theta$.

Solution

There is no identity relating $\cos \theta$ and $\tan \theta$ directly. However, by using the identity relating $\tan \theta$ and $\sec \theta$, we can find $\cos \theta$ by using $\cos \theta = 1/\sec \theta$.

Now

$$\sec^2 \theta = 1 + \tan^2 \theta = 1 + \left(\frac{3}{4}\right)^2 = \frac{25}{16},$$

so $\sec \theta = \pm \frac{5}{4}.$

As θ is an acute angle, we choose the positive sign. Hence

$$\cos \theta = \frac{4}{5}.$$

By using either of the other two identities above, or the identity $\tan \theta = \frac{\sin \theta}{\cos \theta}$, we can similarly show that

$$\sin \theta = \frac{3}{5}.$$

Alternatively, we can obtain these values by considering a right-angled triangle ABC with $BC = 3$ and $AB = 4$ (see Figure 9). Using Pythagoras' theorem, $AC = 5$. Hence

$$\sin \theta = \frac{BC}{AC} = \frac{3}{5} \quad \text{and} \quad \cos \theta = \frac{AB}{AC} = \frac{4}{5}. \quad \square$$

Exercise 5

If θ is the acute angle such that

$$\sin \theta = x,$$

find expressions for $\cos \theta$ and $\tan \theta$ in terms of x .

[Solution on page 63]

You will also have previously met the formulas

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta,$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta,$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta,$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta,$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta},$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.$$

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Subsection 3.4

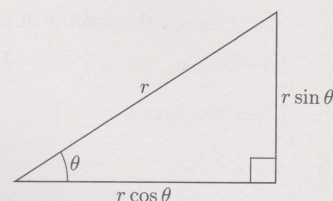


Figure 8

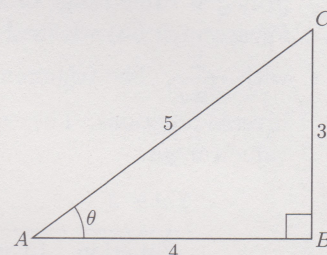


Figure 9 The 3-4-5 triangle.

These identities are all given in Subsection 4.3 of the *Handbook*, although you may find it convenient to memorize the expressions for $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$. By putting $\alpha = 0$ and $\beta = \theta$ in the expressions for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$, it immediately follows that

$$\sin(-\theta) = -\sin \theta \quad \text{and} \quad \cos(-\theta) = \cos \theta,$$

as we can see from the graphs of the sine and cosine functions shown in Figures 3 and 4.

Exercise 6

By putting $\alpha = \frac{1}{3}\pi$ and $\beta = \frac{1}{4}\pi$ in the formula for $\sin(\alpha - \beta)$, show that

$$\sin \frac{1}{12}\pi = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

[Solution on page 63]

Supplementary Exercise 3

Show that the function $A \cos(\omega t + \phi)$ (where A , ω and ϕ are constants and t is a variable) can be written in the form $B \cos \omega t + C \sin \omega t$, giving expressions for B and C in terms of A and ϕ .

[Solution on page 77]

Exercise 7

By putting $\beta = \alpha$ in the formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, prove the identities

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\text{and} \quad \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha.$$

[Solution on page 63]

The identities derived in the above exercise can be useful in integration, as you will see in the audio-tape section associated with Section 4. Again, these results are quoted in Subsection 4.3 of the *Handbook*, so it is not necessary to remember them. However, it is important that you make yourself familiar with this section of your *Handbook* as it contains many useful formulas.

Exercise 8

Use the formula $\cos 2\alpha = 1 - 2 \sin^2 \alpha$ to find the value of $\sin \frac{1}{8}\pi$. (An expression involving square roots is sufficient—there is no need to evaluate the answer as a decimal.)

Exercise 9

By putting $\beta = 2\alpha$ in the formula for $\sin(\alpha + \beta)$, show that

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha.$$

[Solutions on page 63]

You will meet an alternative method of proving this result in Unit 5.

Supplementary Exercise 4

Express $\cos 3\alpha$ in terms of $\cos \alpha$.

[Solution on page 77]

3 Calculus I: Differentiation

3.1 Notation

You will be familiar with the idea of variables connected by an equation. For instance, if an object moves in a straight line with an initial velocity of 2 m s^{-1} and a constant acceleration of 10 m s^{-2} , then the distance it has travelled, x metres, after a time t seconds is given by the equation

$$x = 2t + 5t^2.$$

The feature of this equation that concerns us here is that the left-hand side consists of the single term x and the right-hand side involves only t and not x . The equation tells us that x depends on t , and it enables us to calculate the appropriate value of x as soon as a value of t is given. When two variables x and t are related by any equation in this way, we say that x is a **function** of t . We call x the **dependent variable** and t the **independent variable**.

Since x is a function of t and $x = 2t + 5t^2$, it is logical to say also that $2t + 5t^2$ is a function of t . Indeed, the term ‘a function of t ’ is used to mean any expression whose numerical value is determined by the value of t : for example t^2 , $\sin t$ and e^t are all functions of t . When one variable, say x , is a function of another, say t , we use the notation $x(t)$ to stand for the function of t . So in the above example we can write

$$x(t) = 2t + 5t^2.$$

This notation is particularly convenient if we wish to find the distance travelled after a certain time. For example, the distance travelled after 3 seconds is

$$x(3) = 2 \times 3 + 5 \times 3^2 = 51 \quad (\text{metres}).$$

The numerical value of a function at a particular value of the independent variable is called the **value** of the function. So, for example, the value of the function $x(t)$ at $t = 3$ is $x(3) = 51$.

In mathematics, the most common symbols used for the independent and dependent variables are x and y respectively (and not t and x as above). The symbols t (for time) and x (for distance or position) are commonly used in mechanics, however. When a function is being thought of as a mathematical object we often use the letter f for it (sometimes g or h), and we write $f(x)$ for the result of applying the function f to the number x . So if a variable y is a function of x we should write

$$y = f(x).$$

Our earlier example, where we used the same symbol for the dependent variable and the function, is an abuse of this notation. However, this usage is particularly convenient when it comes to interpreting mathematical equations physically. For instance, in our example it is clear that the derivative

$$x'(t) = 2 + 10t$$

is an expression for the rate of change of the variable x with respect to the variable t , that is, an expression for the velocity. This would have been masked if we had used the strictly correct mathematical notation

$$f'(t) = 2 + 10t.$$

However, there are dangers in using this more informal notation. In *Unit 4* you will see that the velocity v of an object can be thought of either as a function of time t or as a function of position x . So it is not immediately clear whether $v(2)$ means the velocity at time $t = 2$ or the velocity at position $x = 2$.

In *MST204* we shall use differing notations for the **derivative** of a function $y = f(x)$, depending on the context. In the function notation a dash is used to denote a derivative (as above), so that if $f(x) = \sin x$ we have

$$f'(x) = \cos x.$$

In the Leibniz notation the derivative of a variable y with respect to a variable x is denoted by $\frac{dy}{dx}$ (in the main text this is often printed as dy/dx to save space).

For example, if $y = \log_e x$ then

$$\frac{dy}{dx} = \frac{1}{x}.$$

As an extension of this notation, the process of differentiation with respect to x is denoted by the symbol $\frac{d}{dx}$ written to the left of the expression or variable to be differentiated. So

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}.$$

Read $\frac{d}{dx}$ as 'differentiating what follows with respect to x ', or as the command 'differentiate'.

Various mixtures of the function and Leibniz notations are sometimes used. So the derivative of a function $f(x)$ could be denoted by $\frac{df}{dx}$, and the function which is the derivative of a variable y with respect to a variable x might be written as $\frac{dy}{dx}(x)$. This latter notation is especially useful if we want the value of the derivative $\frac{dy}{dx}$ at a particular value of the variable x . So, for example, if $y = x^3 + 2x^2$, then

$$\frac{dy}{dx} = 3x^2 + 4x$$

$$\text{and } \frac{dy}{dx}(2) = 3 \times 2^2 + 4 \times 2 = 20.$$

When the independent variable is x , the derivative of a dependent variable is often denoted by a dash. For instance, if $y = x^{-2}$, we could write

$$y' = -2x^{-3} = -\frac{2}{x^3}.$$

Similarly, when the independent variable is time, usually denoted by t , we often use a dot to denote differentiation. This notation is prevalent in mechanics. So if the velocity v of an object at time t is given by the equation $v = \cos t$, we should have

$$\dot{v} = -\sin t.$$

The derivative of a derivative is called a **second derivative**. For example, the second derivative of the function $f(x)$ is the derivative of $f'(x)$ and is denoted by $f''(x)$. In terms of variables, the second derivative of a variable y with respect to a variable x is

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2 y}{dx^2} = y''.$$

If the independent variable is t , the second derivative of $z = z(t)$ is

$$\frac{d}{dt} \left(\frac{dz}{dt} \right) = \frac{d^2 z}{dt^2} = \ddot{z}.$$

Third and **higher derivatives** are defined and written analogously. However, more than three dashes is cumbersome and so the fourth derivative of the function $f(x)$ is denoted by $f^{(4)}(x)$ rather than $f''''(x)$, and the n th derivative by $f^{(n)}(x)$.

Exercise 1

Calculate the following.

- (i) $f'(x)$, where $f(x) = 2 \sin x + \cos x$ (ii) $\frac{dy}{dx}$, where $y = \sqrt{x} (= x^{1/2})$
 (iii) $v'(\theta)$, where $v = \tan \theta + \sec \theta$ (iv) $\dot{x}$, where $x = t^2 + 2e^t$
 (v) $f'(3)$, where $f(x) = 3x^2 - 4x + 1$ (vi) $\frac{dy}{dt}(0)$, where $y = 3 \tan t - 2 \sin t$
 (vii) $\frac{d^2z}{dx^2}$, where $z = \log_e x$ (viii) $f^{(6)}(\frac{1}{4}\pi)$, where $f(t) = 2 \sin t$

[Note: A list of standard derivatives is given in Subsection 6.3 of the *Handbook*.]

[Solution on page 64]

Supplementary Exercise 1

Calculate the following.

- (i) z' , where $z = \frac{3}{x^3} (= 3x^{-3})$
 (ii) $\frac{d}{dt}(2 + 3 \sin t)$
 (iii) $\dot{x}(\frac{1}{2}\pi)$, where $x = 4 \cos t + 3 \sin t$

[Solution on page 77]

3.2 Differentiation

Differentiation obeys the following rules. Here a and b are constants and u and v are functions of the variable x .

Linearity rule

$$\frac{d}{dx}(au + bv) = a \frac{du}{dx} + b \frac{dv}{dx}$$

Product rule

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u \frac{dv}{dx}$$

Quotient rule

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{\frac{du}{dx}v - u \frac{dv}{dx}}{v^2}$$

Chain rule, composite rule or 'function of a function' rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

A more common way to write this is

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx},$$

where $u = g(x)$ and $y = f(u) (= f(g(x)))$.

Example 1

Use the chain rule to differentiate the following functions.

- (i) $y = \sin(x^2)$ (ii) $y = \sin^2 x$

M101 Block III Unit 1
 Sections 1.1, 1.2 and 1.4
M101 Block III Unit 2
 Sections 2.2 and 2.4

MS284 Unit 9
 Sections 1, 2 and 4
MS284 Unit 10
 Sections 2 and 4

TM282 Unit 5
 Sections 3 and 5
TM282 Unit 7
 Section 3

Solution

- (i) The function
- $y = \sin(x^2)$
- is a 'function of a function' where

$$y = \sin u \quad \text{and} \quad u = x^2.$$

Now

$$\frac{dy}{du} = \cos u = \cos(x^2) \quad \text{and} \quad \frac{du}{dx} = 2x.$$

So

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\cos(x^2)) \times (2x) = 2x \cos(x^2).$$

A function of a function is an alternative name for a composite function.

- (ii) The function
- $y = \sin^2 x$
- is the composite function

$$y = u^2, \quad \text{where} \quad u = \sin x.$$

Now

$$\frac{dy}{du} = 2u = 2 \sin x \quad \text{and} \quad \frac{du}{dx} = \cos x.$$

So

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 2 \sin x \cos x.$$

[Note: You should aim to learn to use the chain rule for composite functions without explicitly writing down the inner function $u = g(x)$, if you cannot do so already. You will soon be able to do so, after a little practice. One way of proceeding is to spot how the function may be expressed as a composite $f(g(x))$, write down f' , putting $g(x)$ as its argument, and then remember to multiply by the 'compensating factor' $g'(x)$.] $\square$

Exercise 2

Differentiate the following functions by using the chain rule.

- (i) $y = \sin(2x + \frac{1}{5}\pi)$ (ii) $s = e^{5t}$ (iii) $y = \arcsin(5x)$
 (iv) $z = (3x^2 + 1)^3$ (v) $z = \log_e(y^3 + 1)$ (vi) $y = \sqrt{4 - x^2}$

[Solution on page 64]

Supplementary Exercise 2

Differentiate the following functions.

- (i) $x = \log_e(3t + 4)$ (ii) $y = \arctan(\frac{1}{2}x)$
 (iii) $y = \exp(t^2)$ (iv) $x = \tan(3 - \frac{1}{2}t^2)$

[Solution on page 77]

Exercise 3

Differentiate the following functions. (These differentiations will involve the use not just of the chain rule, but of the other rules, and possibly more than one at a time.)

- (i) $y = t \sin t$ (ii) $u = \frac{\sin \theta}{\cos \theta}$ (iii) $y = \sec\left(\frac{x}{x^2 + 1}\right)$
 (iv) $x = \sin(t \sin t)$ (v) $x = e^{\sin y} \tan y$

[Solution on page 64]

Supplementary Exercise 3

Differentiate the following functions.

- (i) $x = te^{3t}$ (ii) $f(x) = \frac{\log_e x}{x^2 + 1}$ (iii) $g(t) = t \exp(t^2)$

[Solution on page 77]

When e is raised to the power of a complicated expression, as here, we often write $\exp\{f(x)\}$ rather than $e^{f(x)}$ (since the former is easier to read). So part (v) could be written $x = \exp\{\sin y\} \tan y$.

3.3 Implicit differentiation

Exercise 4

(i) Use the chain rule to differentiate the following.

$$(a) \ y = \sin(x^2) \quad (b) \ y = \sin(x^3 + 1) \quad (c) \ y = \sin(e^x)$$

What is the common pattern to your answers to parts (a) to (c)?

(ii) Use the chain rule to find the derivative of $y = \sin(u(x))$, where $u(x)$ is an (unknown) function of x , giving your answer in terms of u and du/dx .

[Solution on page 64]

Suppose, for instance, that we wish to find dy/dx when the variables x and y satisfy the relationship

$$y + \sin y = x + e^x.$$

It is impossible to derive from this an equation $y = f(x)$ which can be differentiated in the normal way. However, in this case, each term can be differentiated with respect to x as it stands, to give

$$\begin{aligned} \frac{d}{dx}(y) + \frac{d}{dx}(\sin y) &= \frac{d}{dx}(x) + \frac{d}{dx}(e^x), \\ \text{i.e.} \quad \frac{dy}{dx} + \frac{d}{dx}(\sin y) &= 1 + e^x. \end{aligned} \quad (1)$$

Just as in Exercise 4, we can use the chain rule to find the derivative of $\sin y$, even though we do not know explicitly the function $y = y(x)$. This gives

$$\frac{d}{dx}(\sin y) = \frac{d}{dy}(\sin y) \frac{dy}{dx} = \cos y \frac{dy}{dx}.$$

Substituting this in Equation (1) finally leads to

$$\begin{aligned} \frac{dy}{dx} + \cos y \frac{dy}{dx} &= 1 + e^x, \\ \text{or} \quad \frac{dy}{dx} &= \frac{1 + e^x}{1 + \cos y}. \end{aligned}$$

We have to leave the result in this form because we cannot express y in terms of x . The method used above is called **implicit differentiation**.

Exercise 5

Simplify the following, where y is an unknown function of x , leaving your answers as expressions involving x , y and $\frac{dy}{dx}$.

$$\begin{aligned} (i) \quad \frac{d}{dx}(y^2) \quad (ii) \quad \frac{d}{dx}(\tan y) \quad (iii) \quad \frac{d}{dx}(\log_e y) \quad (iv) \quad \frac{d}{dx}(\log_e(\sin y)) \\ (v) \quad \frac{d}{dx}(x^2 y) \quad (vi) \quad \frac{d}{dx}(y^2 e^x) \quad (vii) \quad \frac{d}{dx}(\sin(x^2 + y^2)) \end{aligned}$$

[Solution on page 64]

Supplementary Exercise 4

Simplify the following, where y is an unknown function of x , leaving your answers as expressions involving x , y and $\frac{dy}{dx}$.

$$(i) \quad \frac{d}{dx}(\cos(y^2)) \quad (ii) \quad \frac{d}{dx}(\log_e(x + e^y)) \quad (iii) \quad \frac{d}{dx}(\tan(xy))$$

[Solution on page 77]

Exercise 6

Find the slope of the tangent to the ellipse

$$x^2 + 4y^2 = 8$$

at the point (2, 1) by

- expressing y as a function of x and then differentiating this function,
- differentiating the equation of the ellipse directly, using implicit differentiation.

Exercise 7

Find the slope of the tangent to the curve

$$x^3 + x^2y + y^3 = 1$$

at the point $(-1, 1)$.

[Solutions on page 65]

3.4 Extrema of functions

In mathematical models of real situations, as well as in curve sketching, we are often interested in finding **local maxima** and **local minima** of a function $f(x)$. At a point which is a local maximum, the function has a greater value than at points immediately on either side, and the derivative $f'(x)$ changes from positive, through zero, to negative. Similarly, at a point which is a local minimum, the function has a smaller value than at points immediately on either side, and the derivative changes from negative, through zero, to positive. Thus at a local maximum or a local minimum, the derivative of the function is zero. (Note that the overall or global maximum or minimum of a function in a given range may not occur at a local maximum or minimum but at an extreme point of the range, as shown in Figure 3; here, the derivative may not be zero.)

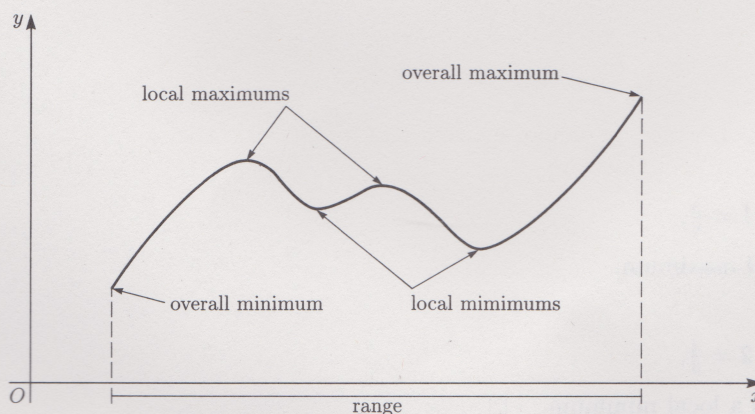


Figure 3

Rather than using the change in sign of the first derivative to decide if a stationary point is a local maximum or a local minimum, it is sometimes more convenient to use the second derivative. If $f''(x) > 0$ at a stationary point, then it is a local minimum; if $f''(x) < 0$, then it is a local maximum. However, if $f''(x) = 0$ at a stationary point, then the point might be a local maximum, a local minimum or a point of inflection with a horizontal tangent. (The functions $y = x^4$, $y = -x^4$ and $y = x^3$ at the point $(0, 0)$ are examples of these three types of behaviour.)

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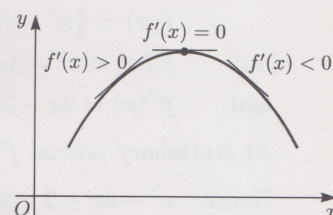


Figure 1 Local maximum.

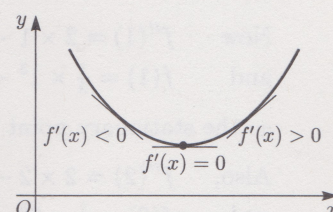


Figure 2 Local minimum.

A point which is either a local maximum or a local minimum is called a **local extremum**.

A **stationary point** is a point at which $f'(x) = 0$.

The procedure for finding and classifying stationary points of a function $y = f(x)$ is summarized below.

- (a) Find $f'(x)$ and $f''(x)$.
- (b) Solve $f'(x) = 0$ to find the x -coordinates of the stationary points.
- (c) Evaluate $f''(x)$ at the stationary points.
 - (i) If $f''(x) < 0$, the point is a local maximum.
 - (ii) If $f''(x) > 0$, the point is a local minimum.
 - (iii) If $f''(x) = 0$, find the sign of $f'(x)$ immediately to the left and right of the stationary point.

Sign to left	Sign to right	Type of point
+	−	Local maximum
−	+	Local minimum
+	+	Point of inflection with horizontal tangent
−	−	

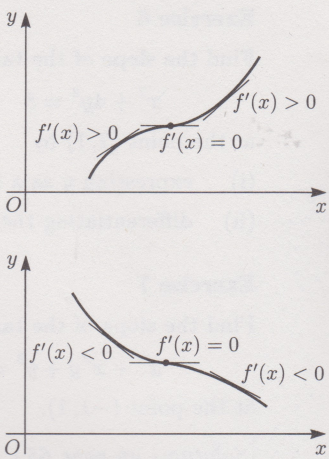


Figure 4 Points of inflection with horizontal tangent.

Example 2

Find and classify the stationary points of the function

$$y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x.$$

Solution

The function is $y = f(x)$ where

$$f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x.$$

So $f'(x) = x^2 - 3x + 2$

and $f''(x) = 2x - 3.$

At stationary points, $f'(x) = 0.$

Hence $x^2 - 3x + 2 = 0,$

i.e. $(x - 1)(x - 2) = 0.$

So $x = 1$ or $x = 2.$

Now $f''(1) = 2 \times 1 - 3 < 0$

and $f(1) = \frac{1}{3} \times 1^3 - \frac{3}{2} \times 1^2 + 2 \times 1 = \frac{5}{6},$

so the stationary point $(1, \frac{5}{6})$ is a local maximum.

Also, $f''(2) = 2 \times 2 - 3 > 0$

and $f(2) = \frac{1}{3} \times 2^3 - \frac{3}{2} \times 2^2 + 2 \times 2 = \frac{2}{3},$

therefore the stationary point $(2, \frac{2}{3})$ is a local minimum. □

Exercise 8

Find and classify the stationary points of the function

$$y = 2x^3 - 3x^2 - 12x + 6.$$

[Solution on page 65]

Supplementary Exercise 5

Find and classify the stationary points of the function

$$y = 3x^4 + 8x^3 - 48x^2 + 200.$$

[Solution on page 77]

Exercise 9

A square sheet of metal of side 60 cm has squares of side x cm cut from each corner, as shown in Figure 5. The remaining piece is then folded to make an open box, as shown in Figure 6.

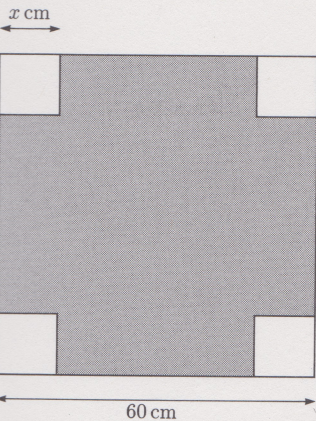


Figure 5

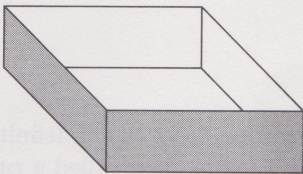


Figure 6

- (i) Derive an expression for the volume of the box.
- (ii) Find the value of x for which the volume of the box is a maximum, and calculate the maximum volume.

[Solution on page 65]

4 Calculus II: Integration

4.1 Standard integrals and integration by guesswork

When you are required to differentiate a function, it is usually not too difficult to decide upon the method or methods to use. Unfortunately, this is not the case for integration, and deciding on the method of integration is often a matter of experience and sometimes even trial and error. The *MST204 Handbook* includes tables of integrals in Subsection 7.1, and you may like to have the *Handbook* open at these tables as you study this section. When confronted with an integral, a natural first step is to consult these tables.

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Sections 3.2 and 3.3
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Exercise 1

Use the tables of standard integrals to evaluate the following.

- (i) $\int \cos 2x \, dx$
- (ii) $\int \frac{1}{\sqrt{9-x^2}} \, dx$
- (iii) $\int_{-1}^1 \sqrt{4-x^2} \, dx$
- (iv) $\int_0^1 \frac{1}{2z+3} \, dz$
- (v) $\int \sin^2 3\theta \, d\theta$

[Solution on page 66]

Supplementary Exercise 1

Use the tables of standard integrals to evaluate the following.

$$(i) \int \sec^2 \frac{1}{3}x \, dx \quad (ii) \int \frac{1}{y^2 + 4} \, dy \quad (iii) \int_0^{\pi/2} \sin 2x \cos 3x \, dx$$

[Solution on page 77]

If the integral does not appear in the tables of standard integrals, we can often use the **fundamental theorem of calculus** to evaluate it. This theorem states that if we can find a function $F(x)$ such that

$$\frac{dF}{dx} = f(x), \quad (1)$$

$$\text{then} \quad \int f(x) \, dx = F(x) + C,$$

where we have included an arbitrary constant of integration C for the indefinite integral, as is the practice in this course. The function $F(x)$ is often called a **primitive**.

The following technique is useful. Given a function $f(x)$, we guess a possible form for the function $F(x)$ and then possibly refine our guess until Equation (1) is satisfied.

This process, which is sometimes called **integration by guesswork**, should become clear after reading the following two examples. However, if you find these examples or any of the following exercises difficult, you may find it easier to evaluate them by using integration by substitution, which is discussed in the next subsection.

Example 1

Evaluate

$$\int (2x + 3)^4 \, dx.$$

Solution

The integrand is a composite function, namely u^4 where $u = 2x + 3$. We recall the standard derivative

$$\frac{d}{dx}(x^5) = 5x^4,$$

which suggests that a possible first guess for the primitive $F(x)$ is $(2x + 3)^5$. However,

$$\frac{d}{dx}(2x + 3)^5 = 10(2x + 3)^4,$$

which differs from the integrand $f(x) = (2x + 3)^4$ by a multiplicative factor 10. So we adjust our initial guess to give the primitive function

$$F(x) = \frac{1}{10}(2x + 3)^5.$$

Finally, we check that

$$\frac{dF}{dx} = \frac{d}{dx} \left(\frac{1}{10}(2x + 3)^5 \right) = (2x + 3)^4 = f(x),$$

as required. We conclude that

$$\int (2x + 3)^4 \, dx = \frac{1}{10}(2x + 3)^5 + C. \quad \square$$

Example 2

Evaluate

$$\int x^2 \sin(x^3 + 1) \, dx.$$

Solution

Concentrating on the most complicated part of the integrand $f(x) = x^2 \sin(x^3 + 1)$, which is the composite function $\sin(x^3 + 1)$, we recall the standard derivative

$$\frac{d}{dx}(\cos x) = -\sin x.$$

This suggests that our first guess for the primitive $F(x)$ is $-\cos(x^3 + 1)$ (or possibly $+\cos(x^3 + 1)$). However,

$$\frac{d}{dx}(-\cos(x^3 + 1)) = 3x^2 \sin(x^3 + 1),$$

which differs from the integrand $f(x) = x^2 \sin(x^3 + 1)$ by a *constant* multiplicative factor 3. So we adjust our initial guess to give the primitive function

$$F(x) = -\frac{1}{3} \cos(x^3 + 1).$$

Finally, we check that

$$\frac{dF}{dx} = \frac{d}{dx}\left(-\frac{1}{3} \cos(x^3 + 1)\right) = x^2 \sin(x^3 + 1) = f(x),$$

as required, and conclude that

$$\int x^2 \sin(x^3 + 1) dx = -\frac{1}{3} \cos(x^3 + 1) + C. \quad \square$$

As you can see, integration by guesswork depends on the fact that differentiation and integration are inverse operations, apart from an arbitrary constant of integration, and basically uses your ability to differentiate composite functions. If you find any of the following exercises difficult do not worry, as we shall consider an alternative method of evaluation in the next subsection.

Exercise 2

Use integration by guesswork to evaluate the following integrals.

- (i) $\int \cos(2x + 3) dx$ (ii) $\int x e^{x^2} dx$
 (iii) $\int x^2(x^3 + 1)^3 dx$ (iv) $\int x \sqrt{x^2 - 1} dx$

[Solution on page 66]

Supplementary Exercise 2

Use integration by guesswork to evaluate the following integrals.

- (i) $\int (3x + 1)^9 dx$ (ii) $\int \frac{x}{2x^2 + 1} dx$

[Solution on page 78]

4.2 Integration by substitution (Audio-tape Subsection)

If you found Exercise 2 difficult, an alternative approach is to use the technique called **integration by substitution**. This method depends less on inspiration, but is more time-consuming! The technique is outlined on the audio-tape associated with this subsection. From your prerequisite course you may be used to a slightly different approach: our advice is to use the method with which you are most familiar. While listening to the tape, you might find it helpful to have the *Handbook* open at the tables of standard integrals in Subsection 7.1. Note that detailed solutions to the exercises in the audio-tape frames are given immediately after the frames.

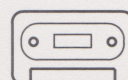
It is important that the difference between the derivative of the guess and the integrand is a constant factor and not a function of x .

M101 Block III Unit 4
Section 4.2

MS284 Unit 12
Section 2

TM282 Unit 11
Section 2

Start the audio-tape when you are ready.



1 The Formula

Integration by substitution:

$$\underbrace{\int f(u(x)) \frac{du}{dx} dx}_{\text{function of } x} = \underbrace{\int f(u) du}_{\text{function of } u}$$

Example

$$\int x \cos(x^2) dx$$

$$\text{Let } u = x^2.$$

Choose function $u(x)$
(substitution)

$$\text{Then } \frac{du}{dx} = 2x,$$

$$\text{so } du = 2x dx.$$

Find du in terms
of x and dx

$$\int x \cos(x^2) dx = \int \frac{1}{2} \cos(x^2) (2x dx)$$

$$= \frac{1}{2} \int \cos u du$$

Write integral in
terms of u and du

$$= \frac{1}{2} \sin u + c$$

Integrate

$$= \frac{1}{2} \sin(x^2) + c.$$

Write answer in
terms of x

2 Exercises

Integrate the following.

Substitution

(i) $\int x^2(x^3+1)^9 dx$ using $u = x^3+1$

(ii) $\int \cos x e^{\sin x} dx$ using $u = \sin x$

(iii) $\int \sin x \cos^2 x dx$ using $u = \cos x$

(iv) $\int \frac{x}{\sqrt{1+x^2}} dx$ using $u = 1+x^2$

3 Choosing a Substitution

$$\int \underbrace{f(u(x))}_{\text{spot the composite}} \frac{du}{dx} dx = \int f(u) du$$

spot the
composite

Try $u =$ inner function
of composite

Example

$$\int x^2 (x^3 + 1)^9 dx.$$

$$\left(\begin{array}{c} \text{inner} \\ \text{function} \end{array} \right)^9 \text{ where inner function } = x^3 + 1,$$

so try substitution $u = x^3 + 1$.

Exercises

Choose a substitution for the following.

(i) $\int x(1+x^2)^2 dx$

(ii) $\int e^x \cos(e^x) dx$

(iii) $\int \frac{x^2}{x^3+1} dx$

(iv) $\int \frac{\cos x}{\sin^2 x} dx \quad (0 < x < \pi)$

4 A Special Case

$$\int \frac{g'(x)}{g(x)} dx$$

Quotient: top is
derivative of bottom

Use substitution $u = g(x)$.

Example

$$\int \frac{2x}{x^2+1} dx$$

$$x^2 + 1 = g(x)$$

$$2x = g'(x)$$

Try $u = x^2 + 1$;

then $\frac{du}{dx} = 2x$

and $du = 2x dx$.

$$= \int \frac{1}{u} du$$

$u > 0$

$$= \log_e u + c = \log_e (x^2 + 1) + c$$

Similarly,
if $g(x) < 0$,

$$\int \frac{g'(x)}{g(x)} dx = \log_e (g(x)) + c$$

5 Exercises

Integrate the following.

(i) $\int \frac{e^x}{e^x + 1} dx$

(ii) $\int \frac{\cos x}{\sin x} dx \quad (0 < x < \pi)$

(iii) $\int \frac{x}{x^2 - 1} dx \quad (x > 1)$

6 Still the Special Case

$$\int \frac{g'(x)}{g(x)} dx: \text{ what if } g(x) < 0?$$

Substitution $u = -g(x)$ leads to

$$\frac{du}{dx} = -g'(x)$$

$$\int \frac{1}{u} du, \text{ where } u > 0,$$

$$= \log_e u + c = \log_e (-g(x)) + c.$$

Example

$$\int \frac{x}{x^2 - 1} dx \quad (-1 < x < 1)$$

$$x^2 - 1 < 0, \text{ so } u = 1 - x^2; \\ \text{then } du = -2x dx.$$

$$= \frac{1}{2} \log_e (1 - x^2) + c$$

7 Definite Integrals: an Example

$$\int_1^3 \frac{2x}{x^2 + 1} dx$$

$$\text{Substitution } u = x^2 + 1 \\ \text{gives } du = 2x dx$$

$$\text{Limits: } x = 1 \xrightarrow{\text{lower}} u = 1^2 + 1 = 2$$

$$x = 3 \xrightarrow{\text{upper}} u = 3^2 + 1 = 10$$

$$\int_1^3 \frac{2x}{x^2 + 1} dx$$

integral in terms of x

$$= \int_2^{10} \frac{1}{u} du$$

in terms of u

$$= [\log_e u]_2^{10} = \log_e 10 - \log_e 2 = \log_e 5$$

8 Exercises

Evaluate the following.

(i) $\int_0^{\pi/3} \frac{\sin x}{\cos x} dx$

(ii) $\int_1^2 x\sqrt{x^2-1} dx$

9 A Harder Example

$$\int \frac{x}{(1+x)^2} dx \quad (x > -1).$$

Let $u = 1+x$ ($u > 0$);

then $\frac{du}{dx} = 1$ and $du = dx$.

Choose u

Find du

need x in terms of u :
 $x = u - 1$

Substitute

$$\int \frac{x}{(1+x)^2} dx = \int \frac{u-1}{u^2} du$$

Integrate

$$= \int \left(\frac{1}{u} - \frac{1}{u^2} \right) du$$

$$= \log_e u + \frac{1}{u} + c$$

In terms of x

$$= \log_e (1+x) + \frac{1}{1+x} + c$$

Exercises

Integrate the following.

(i) $\int x\sqrt{x-1} dx \quad (x > 1)$

(ii) $\int x(2x+1)^9 dx$

10 Trigonometric Substitution

If integral involves

- $\sqrt{a^2 - x^2}$ ($-a < x < a$), try $x = a \sin u$,

$$\text{then } \sqrt{a^2 - x^2} = a \sqrt{1 - \sin^2 u} = a \cos u;$$

- $a^2 + x^2$, try $x = a \tan u$,

$$\text{then } a^2 + x^2 = a^2 (1 + \tan^2 u) = a^2 \sec^2 u.$$

Example

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx \quad (-a < x < a).$$

$$\text{Let } x = a \sin u;$$

$$u = \arcsin \left(\frac{x}{a} \right)$$

$$\text{then } \frac{dx}{du} = a \cos u \quad \text{and} \quad dx = a \cos u \, du.$$

$$\text{Also } \sqrt{a^2 - x^2} = a \cos u.$$

$$\begin{aligned} \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \int \frac{1}{a \cos u} (a \cos u \, du) \\ &= \int 1 \, du = u + c \\ &= \arcsin \left(\frac{x}{a} \right) + c \end{aligned}$$

11 Exercises

Integrate the following.

$$(i) \int \frac{1}{\sqrt{4 - x^2}} dx \quad (-2 < x < 2)$$

$$(ii) \int \frac{5}{9 + x^2} dx$$

$$(iii) \int \frac{1}{(1 + x^2)^{3/2}} dx$$

$$(iv) \int \sqrt{1 - x^2} dx \quad (-1 < x < 1)$$

Solutions to Exercises in the Tape Frames

Frame 2

- (i) To evaluate
- $\int x^2(x^3 + 1)^9 dx$
- , let
- $u = x^3 + 1$
- .

Then $\frac{du}{dx} = 3x^2$, so that $du = 3x^2 dx$.

Hence

$$\begin{aligned}\int x^2(x^3 + 1)^9 dx &= \int \frac{1}{3}((x^3 + 1)^9)(3x^2 dx) \\ &= \frac{1}{3} \int u^9 du \\ &= \frac{1}{30} u^{10} + C \\ &= \frac{1}{30} (x^3 + 1)^{10} + C.\end{aligned}$$

- (ii) To evaluate
- $\int \cos x e^{\sin x} dx$
- , let
- $u = \sin x$
- .

Then $\frac{du}{dx} = \cos x$, so that $du = \cos x dx$.

Hence

$$\begin{aligned}\int \cos x e^{\sin x} dx &= \int (e^{\sin x})(\cos x dx) \\ &= \int e^u du \\ &= e^u + C \\ &= e^{\sin x} + C.\end{aligned}$$

- (iii) To evaluate
- $\int \sin x \cos^2 x dx$
- , let
- $u = \cos x$
- .

Then $\frac{du}{dx} = -\sin x$, so that $du = -\sin x dx$.

Hence

$$\begin{aligned}\int \sin x \cos^2 x dx &= - \int ((\cos x)^2)(-\sin x dx) \\ &= - \int u^2 du \\ &= -\frac{1}{3} u^3 + C \\ &= -\frac{1}{3} \cos^3 x + C.\end{aligned}$$

- (iv) To evaluate
- $\int \frac{x}{\sqrt{1+x^2}} dx$
- , let
- $u = 1 + x^2$
- .

Then $\frac{du}{dx} = 2x$, so that $du = 2x dx$.

Hence

$$\begin{aligned}\int \frac{x}{\sqrt{1+x^2}} dx &= \frac{1}{2} \int \left(\frac{1}{(1+x^2)^{1/2}} \right) (2x dx) \\ &= \frac{1}{2} \int \frac{1}{u^{1/2}} du \\ &= \frac{1}{2} \int u^{-1/2} du \\ &= u^{1/2} + C \\ &= \sqrt{1+x^2} + C.\end{aligned}$$

Frame 3

Suitable substitutions for the integrals are as follows.

- (i)
- $u = 1 + x^2$
- (ii)
- $u = e^x$
- (iii)
- $u = x^3 + 1$
- (iv)
- $u = \sin x$

Frame 5

- (i) To evaluate $\int \frac{e^x}{e^x + 1} dx$, let $u = e^x + 1$, which we note is always positive.

Then $\frac{du}{dx} = e^x$, so that $du = e^x dx$.

Hence

$$\begin{aligned}\int \frac{e^x}{e^x + 1} dx &= \int \frac{1}{e^x + 1} (e^x dx) \\ &= \int \frac{1}{u} du \quad (\text{where } u > 0) \\ &= \log_e u + C \\ &= \log_e (e^x + 1) + C.\end{aligned}$$

- (ii) To evaluate $\int \frac{\cos x}{\sin x} dx$ ($= \int \cot x dx$), let $u = \sin x$, which is positive for $0 < x < \pi$.

Then $\frac{du}{dx} = \cos x$, so that $du = \cos x dx$.

Hence

$$\begin{aligned}\int \frac{\cos x}{\sin x} dx &= \int \frac{1}{\sin x} (\cos x dx) \\ &= \int \frac{1}{u} du \quad (\text{where } u > 0) \\ &= \log_e u + C \\ &= \log_e (\sin x) + C.\end{aligned}$$

- (iii) To evaluate $\int \frac{x}{x^2 - 1} dx$, let $u = x^2 - 1$, which is positive for $x > 1$.

Then $\frac{du}{dx} = 2x$, so that $du = 2x dx$.

Hence

$$\begin{aligned}\int \frac{x}{x^2 - 1} dx &= \frac{1}{2} \int \frac{1}{x^2 - 1} (2x dx) \\ &= \frac{1}{2} \int \frac{1}{u} du \quad (\text{where } u > 0) \\ &= \frac{1}{2} \log_e u + C \\ &= \frac{1}{2} \log_e (x^2 - 1) + C.\end{aligned}$$

For the case $x < -1$, $x^2 - 1$ is again positive and so we can again use the substitution $u = x^2 - 1$. The working is exactly as above, to obtain the result

$$\int \frac{x}{x^2 - 1} dx = \frac{1}{2} \log_e (x^2 - 1) + C.$$

However, in the case $-1 < x < 1$, $x^2 - 1$ is negative and it is necessary to use the substitution $u = 1 - x^2$, as discussed in Frame 6.

With this substitution $\frac{du}{dx} = -2x$, so that $du = -2x dx$. Hence

$$\begin{aligned}\int \frac{x}{x^2 - 1} dx &= \frac{1}{2} \int \frac{1}{1 - x^2} (-2x dx) \\ &= \frac{1}{2} \int \frac{1}{u} du \quad (\text{where } u > 0) \\ &= \frac{1}{2} \log_e u + C \\ &= \frac{1}{2} \log_e (1 - x^2) + C.\end{aligned}$$

Summarizing the above results,

$$\int \frac{x}{x^2 - 1} dx = \begin{cases} \frac{1}{2} \log_e (x^2 - 1) + C & (x < -1), \\ \frac{1}{2} \log_e (1 - x^2) + C & (-1 < x < 1), \\ \frac{1}{2} \log_e (x^2 - 1) + C & (x > 1). \end{cases}$$

Frame 8

- (i) To evaluate $\int_0^{\pi/3} \frac{\sin x}{\cos x} dx \left(= \int_0^{\pi/3} \tan x dx \right)$, let $u = \cos x$, noting that $u > 0$ for $0 \leq x \leq \frac{1}{3}\pi$.

Then $\frac{du}{dx} = -\sin x$, so that $du = -\sin x dx$.

Furthermore, when $x = 0$, $u = \cos 0 = 1$, and when $x = \frac{1}{3}\pi$, $u = \cos \frac{1}{3}\pi = \frac{1}{2}$.

So

$$\begin{aligned} \int_0^{\pi/3} \frac{\sin x}{\cos x} dx &= - \int_{x=0}^{x=\pi/3} \frac{1}{\cos x} (-\sin x dx) \\ &= - \int_{u=1}^{u=1/2} \frac{1}{u} du \\ &= - [\log_e u]_{u=1}^{u=1/2} \\ &= - (\log_e \frac{1}{2} - \log_e 1) \\ &= -(-\log_e 2 - 0) \\ &= \log_e 2. \end{aligned}$$

- (ii) To evaluate $\int_1^2 x\sqrt{x^2-1} dx$, let $u = x^2 - 1$.

Then $\frac{du}{dx} = 2x$, so that $du = 2x dx$.

When $x = 1$, $u = 1^2 - 1 = 0$, and when $x = 2$, $u = 2^2 - 1 = 3$.

Hence

$$\begin{aligned} \int_1^2 x\sqrt{x^2-1} dx &= \frac{1}{2} \int_{x=1}^{x=2} (x^2-1)^{1/2} (2x dx) \\ &= \frac{1}{2} \int_{u=0}^{u=3} u^{1/2} du \\ &= \left[\frac{1}{3} u^{3/2} \right]_{u=0}^{u=3} \\ &= \frac{1}{3} (3^{3/2} - 0^{3/2}) \\ &= \frac{1}{3} (3\sqrt{3}) \\ &= \sqrt{3}. \end{aligned}$$

Frame 9

- (i) To evaluate $\int x\sqrt{x-1} dx$, let $u = x - 1$, so that $x = u + 1$.

Then $\frac{du}{dx} = 1$, so that $du = dx$.

Hence

$$\begin{aligned} \int x\sqrt{x-1} dx &= \int (u+1)\sqrt{u} du \\ &= \int (u^{3/2} + u^{1/2}) du \\ &= \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C \\ &= \frac{2}{5} (x-1)^{5/2} + \frac{2}{3} (x-1)^{3/2} + C. \end{aligned}$$

- (ii) To evaluate $\int x(2x+1)^9 dx$, let $u = 2x + 1$, so that $x = \frac{1}{2}u - \frac{1}{2}$.

Then $\frac{du}{dx} = 2$, so that $du = 2 dx$.

Hence

$$\begin{aligned}
 \int x(2x+1)^9 dx &= \frac{1}{2} \int \left(\frac{1}{2}u - \frac{1}{2}\right)u^9 du \\
 &= \int \left(\frac{1}{4}u^{10} - \frac{1}{4}u^9\right) du \\
 &= \frac{1}{44}u^{11} - \frac{1}{40}u^{10} + C \\
 &= \frac{1}{44}(2x+1)^{11} - \frac{1}{40}(2x+1)^{10} + C.
 \end{aligned}$$

Frame 11

(i) To evaluate $\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\sqrt{2^2-x^2}} dx$, let $x = 2 \sin u$.

Then $\frac{dx}{du} = 2 \cos u$, so that $dx = 2 \cos u du$.

Hence

$$\begin{aligned}
 \int \frac{1}{\sqrt{4-x^2}} dx &= \int \frac{1}{\sqrt{4(1-\sin^2 u)}} (2 \cos u du) \\
 &= \int \frac{2 \cos u}{2 \cos u} du \\
 &= \int du \\
 &= u + C \\
 &= \arcsin\left(\frac{1}{2}x\right) + C.
 \end{aligned}$$

(ii) To evaluate $\int \frac{5}{9+x^2} dx = \int \frac{5}{3^2+x^2} dx$, let $x = 3 \tan u$.

Then $\frac{dx}{du} = 3 \sec^2 u$, so that $dx = 3 \sec^2 u du$.

Hence

$$\begin{aligned}
 \int \frac{5}{9+x^2} dx &= \int \frac{5}{9(1+\tan^2 u)} (3 \sec^2 u du) \\
 &= \frac{5}{3} \int \frac{\sec^2 u}{\sec^2 u} du \\
 &= \frac{5}{3} \int du \\
 &= \frac{5}{3} u + C \\
 &= \frac{5}{3} \arctan\left(\frac{1}{3}x\right) + C.
 \end{aligned}$$

(iii) To evaluate $\int \frac{1}{(1+x^2)^{3/2}} dx$, let $x = \tan u$.

Then $\frac{dx}{du} = \sec^2 u$, so that $dx = \sec^2 u du$.

Hence

$$\begin{aligned}
 \int \frac{1}{(1+x^2)^{3/2}} dx &= \int \frac{1}{(1+\tan^2 u)^{3/2}} (\sec^2 u du) \\
 &= \int \frac{\sec^2 u}{\sec^3 u} du \\
 &= \int \frac{1}{\sec u} du \\
 &= \int \cos u du \\
 &= \sin u + C \\
 &= \sin(\arctan x) + C \\
 &= \frac{x}{\sqrt{1+x^2}} + C,
 \end{aligned}$$

where the identity

$$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$$

follows from the right-angled triangle in Figure 1 for x positive. In fact, the final answer is also correct for x negative, as can be verified by differentiation.

- (iv) To evaluate $\int \sqrt{1-x^2} dx$, let $x = \sin u$.

Then $\frac{dx}{du} = \cos u$, so that $dx = \cos u du$.

Hence

$$\begin{aligned} \int \sqrt{1-x^2} dx &= \int (\sqrt{1-\sin^2 u}) (\cos u du) \\ &= \int \cos^2 u du \\ &= \frac{1}{2} \int (1 + \cos 2u) du \\ &= \frac{1}{2} u + \frac{1}{4} \sin 2u + C \\ &= \frac{1}{2} u + \frac{1}{2} \sin u \cos u + C \\ &= \frac{1}{2} u + \frac{1}{2} \sin u \sqrt{1-\sin^2 u} + C \\ &= \frac{1}{2} \arcsin x + \frac{1}{2} x \sqrt{1-x^2} + C. \end{aligned}$$

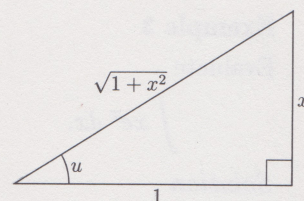


Figure 1

End of solutions to tape frame exercises.

Exercise 3

Use integration by substitution to evaluate the following integrals.

- (i) $\int x(x^2-5)^7 dx$ (ii) $\int \frac{t}{1-t^2} dt \quad (-1 < t < 1)$ (iii) $\int z^3(1-z^4)^6 dz$
 (iv) $\int_0^7 x(x+1)^{1/3} dx$ (v) $\int \frac{1}{(4+x^2)^2} dx \quad (x > 0)$ (vi) $\int_1^2 x\sqrt{x-1} dx$

[Solution on page 66]

Supplementary Exercise 3

Use integration by substitution to evaluate the following integrals.

- (i) $\int x^2 \sqrt{x^3+1} dx \quad (x > 0)$ (ii) $\int te^{t^2} dt$ (iii) $\int \cos \theta \sin^3 \theta d\theta$
 (iv) $\int \theta \cos(\theta^2) d\theta$ (v) $\int \frac{x}{\sqrt{2x-1}} dx \quad (x > \frac{1}{2})$ (vi) $\int_0^{1/2} \sqrt{1-x^2} dx$

[Solution on page 78]

You may be able to evaluate some of these by using integration by guesswork, but do them by substitution for the sake of the practice.

M101 Block III Unit 3
Section 3.4

M101 Block III Unit 4
Section 4.1

MS284 Unit 11
Section 5
MS284 Unit 12
Section 1

TM282 Unit 10
Section 4

4.3 Integration by parts

The final method of integration that we shall consider is **integration by parts**.

However, this is not really a technique for integration, but a method of converting an untenable integral into another integral which we hope can be integrated by using the tables of standard integrals, integration by guesswork or integration by substitution.

In function notation, the formula for integration by parts can be written as

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx,$$

whereas in Leibniz notation it is

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

Unfortunately, the notation for this formula in the literature is far from standard; our advice is to use the notation with which you are familiar.

Example 3

Evaluate

$$\int x e^x dx.$$

Solution

Using Leibniz notation, we choose

$$u = x \quad \text{and} \quad \frac{dv}{dx} = e^x.$$

$$\text{So} \quad \frac{du}{dx} = 1 \quad \text{and} \quad v = e^x.$$

Substituting in the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx,$$

we obtain

$$\begin{aligned} \int x e^x dx &= x e^x - \int (e^x)(1) dx \\ &= x e^x - \int e^x dx \\ &= x e^x - e^x + C. \quad \square \end{aligned}$$

Note that we do not include a constant of integration in v .

Before we can use integration by parts, we first have to decide how to write the integrand as the product of two functions, namely u and $\frac{dv}{dx}$, although this is usually obvious. Secondly, we have to decide which part of the product to call u and which $\frac{dv}{dx}$. This is important because if you make the wrong choice you could make the integral more difficult rather than simpler! A good rule of thumb is to choose as u the function in the product which occurs first in the following list:

- logarithm function,
- polynomial function,
- exponential function,
- sine or cosine function.

Obviously, you need to ensure that you can integrate your choice of $\frac{dv}{dx}$ in order to be able to find the function v .

Exercise 4

Use integration by parts to evaluate the following integrals.

$$(i) \int x \log_e x dx \quad (ii) \int x^2 e^x dx \quad (iii) \int \log_e x dx$$

[Hint: You will need to use integration by parts twice to evaluate the integral in part (ii). In part (iii) first write the integrand as the product $(1)(\log_e x)$.]

[Solution on page 67]

Supplementary Exercise 4

Use integration by parts to evaluate the following integrals.

$$(i) \int x e^{-2x} dx \quad (ii) \int x \cos x dx$$

[Solution on page 78]

In Exercise 4(ii) we have seen that sometimes it is necessary to use integration by parts more than once. For example, to evaluate $\int x^n e^x dx$, $\int x^n \sin x dx$ or $\int x^n \cos x dx$, where n is a positive integer, it is necessary to integrate by parts n times before we obtain a standard integral. In the following exercise we shall see one further case where we use integration by parts twice.

Exercise 5

By using integration by parts twice, evaluate

$$I = \int e^x \sin x \, dx.$$

[Solution on page 68]

The formula for using integration by parts for definite integrals is

$$\int_a^b u \frac{dv}{dx} dx = [uv]_{x=a}^{x=b} - \int_a^b v \frac{du}{dx} dx.$$

As you will see in Exercise 7, it is sensible to substitute the limits $x = a$ and $x = b$ in the uv term at each stage when using the integration by parts formula a number of times in order to evaluate a definite integral.

Exercise 6

Use integration by parts to evaluate the definite integral

$$\int_0^{\pi/2} x \sin 2x \, dx.$$

Exercise 7

Consider the definite integral

$$I_n = \int_0^\infty x^n e^{-x} \, dx.$$

(i) Evaluate I_0 , using the fact that $e^{-x} \rightarrow 0$ as $x \rightarrow \infty$.

(ii) Use integration by parts to show that

$$I_n = nI_{n-1} \quad (n = 1, 2, 3, \dots).$$

(iii) Use the results of parts (i) and (ii) to evaluate I_n ($n = 1, 2, 3, \dots$).

[Solutions on page 68]

We shall see another method of evaluating this integral in Unit 5.

Strictly speaking, we should say

$$I_n = \lim_{k \rightarrow \infty} \int_0^k x^n e^{-x} \, dx.$$

5 Graphs and approximations

5.1 The straight line

The most general equation of a **straight line** in the (x, y) -plane is the linear equation

$$ax + by = c,$$

where at least one of the coefficients a and b is non-zero. Apart from lines parallel to the y -axis, this equation is often written in the form

$$y = mx + c,$$

where m is the slope or gradient of the line and c is the y -intercept, i.e. the value of y at the point where the line crosses the y -axis (see Figure 1).

Countdown to Mathematics
Subsections 3.3 and 7.1

TM282 Unit 1
Section 3

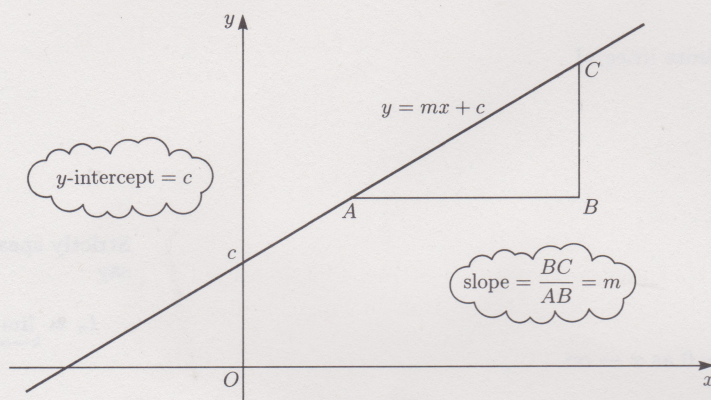


Figure 1 The straight line $y = mx + c$.

Example 1

Find the equation of the straight line which passes through the points $(1, 2)$ and $(2, 5)$. Deduce the slope of this line and its x - and y -intercepts.

Solution

Let the equation of the straight line be

$$y = mx + c.$$

The points $(1, 2)$ and $(2, 5)$ are on the straight line, hence

$$2 = m + c$$

and $5 = 2m + c.$

Solving these equations, we obtain

$$m = 3 \quad \text{and} \quad c = -1.$$

Hence the straight line is

$$y = 3x - 1.$$

The slope of the line is

$$m = 3.$$

The line crosses the y -axis where $x = 0$, so

$$y = -1.$$

The y -intercept is therefore $c = -1$.

The line crosses the x -axis where $y = 0$, so

$$3x - 1 = 0,$$

i.e. $x = \frac{1}{3},$

which is the x -intercept. $\square$

Exercise 1

- (i) Find the
- x
- and
- y
- intercepts of the straight line

$$2x + 3y = 3.$$

Hence draw the graph of this line.

- (ii) Draw the graph of the line

$$y = \frac{1}{2}x$$

by finding the coordinates of two points on the line.

Exercise 2Find the equation of the straight line with slope -2 which passes through the point $(1, 1)$.

[Solutions on page 68]

Supplementary Exercise 1Find the equation of the straight line with slope 3 which passes through the point $(-1, 2)$.
Find the x - and y -intercepts of this line.

[Solution on page 79]

5.2 Curve sketching

This subsection is intended to help you to sketch a curve without just plotting points on the curve. Often, a knowledge of the basic shapes of the graphs in Subsection 5.3 of the *Handbook* is sufficient.

Exercise 3

Sketch the graphs of the following functions.

- (i)
- $y = 2 \sin 3x$
- (ii)
- $y = e^{x/8} \cos x$

[Solution on page 69]

Even if you know the sort of shape to expect, you may need more information in order to sketch a particular graph. For instance, you may need to know the coordinates of the local maxima and minima, and where the graph crosses the axes. A possible procedure for sketching a graph is given below.

- Find where the curve crosses the x -axis and y -axis, if at all.
- Determine how y behaves when x is very large and positive and when x is very large and negative.
- Look for any values of x for which the function is undefined, and examine the behaviour of y near these values of x .
- Find and classify any stationary points of the function.
- Transfer the above information to a diagram, and use this to draw the sketch graph.

M101 Block III Unit 2
Section 2.3

MS284 Unit 10
Section 3

See Subsection 3.4 of this unit.

Example 2

Sketch the graph of

$$y = \frac{x+1}{x+2}.$$

Solution

When $x = 0$, $y = \frac{1}{2}$, and so the curve crosses the y -axis at $(0, \frac{1}{2})$. When $y = 0$, $x = -1$, and so the graph crosses the x -axis at $(-1, 0)$.

For very large x ,

$$y \simeq \frac{x}{x} = 1.$$

In fact, when x is large and positive, y is just less than 1, whereas when x is large and negative, y is just greater than 1.

The function is undefined when $x = -2$. When x is just less than -2 , y is large and positive, whereas when x is just greater than -2 , y is large and negative.

$$\frac{dy}{dx} = \frac{(1) \times (x+2) - (x+1) \times (1)}{(x+2)^2} = \frac{1}{(x+2)^2}.$$

This cannot be equal to zero for any values of x and so the function has no stationary points.

Collecting together the above information, we conclude that the sketch of the curve is as shown in Figure 2.

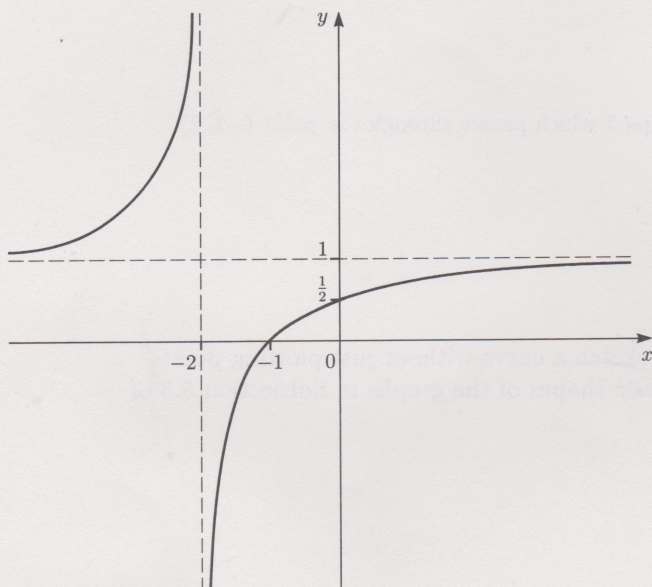


Figure 2 Sketch of the curve $y = \frac{x+1}{x+2}$. $\square$

Example 3

Sketch the graph of

$$y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x.$$

Solution

When $x = 0$, $y = 0$, and so the curve crosses the y -axis at the origin.

When $y = 0$,

$$\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x = 0,$$

$$\text{or } x \left(\frac{1}{3}x^2 - \frac{3}{2}x + 2 \right) = 0.$$

$$\text{So } x = 0 \quad \text{or} \quad 2x^2 - 9x + 12 = 0.$$

The quadratic has no (real) solutions and so the curve crosses the x -axis only at the origin.

For large x ,

$$y \simeq \frac{1}{3}x^3.$$

In particular, when x is large and positive, y is also large and positive, whereas when x is large and negative, y is large and negative.

The function is defined for all values of x .

Using the solution to Example 2 of Subsection 3.4, the function has a local maximum at $(1, \frac{5}{6})$ and a local minimum at $(2, \frac{2}{3})$.

Collecting together the above information, we arrive at the sketch of the curve shown in Figure 3.

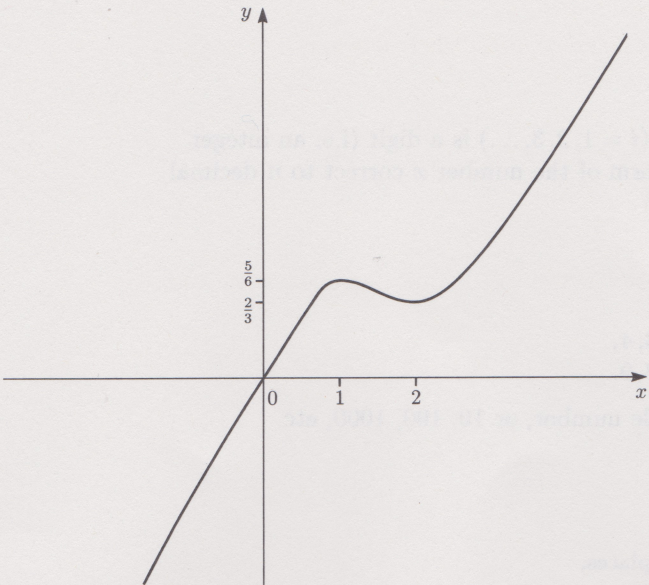


Figure 3 Sketch of the curve $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$. □

Exercise 4

Sketch the graph of the function

$$y = 2x^3 - 3x^2 - 12x + 6.$$

[You will find the answer to Exercise 8 of Section 3 useful. Do not worry if you cannot find the points where the curve crosses the x -axis!]

Exercise 5

Sketch the graph of the function

$$y = \frac{x}{(x - 1)^2}.$$

[Solutions on page 69]

Supplementary Exercise 2

Sketch the graph of the function

$$y = 3x^4 + 8x^3 - 48x^2 + 200.$$

[You will find the answer to Supplementary Exercise 5 of Section 3 useful.]

Supplementary Exercise 3

Sketch the graph of the function

$$y = \frac{x^2}{x + 1}.$$

[Solutions on page 79]

5.3 Rounding and accuracy

If I use my calculator to write $\frac{1}{3}$ as a decimal, I obtain 0.333 333 333, which is only an approximation to the fraction, although a very good one! Here the fraction has been **rounded** to nine decimal places, and we say that the decimal is **correct to nine decimal places**. We can, of course, round to any number of decimal places. For example, $\frac{1}{3}$ correct to three decimal places is 0.333, whereas $\frac{2}{3}$ is 0.667. In this last example note that $\frac{2}{3}$ is closer to the value 0.667 than it is to 0.666. Formally, the rule is as follows.

Countdown to Mathematics
Subsection 1.1

M101 Block V Unit 1
Section 1.5

MS284 Unit 14
Subsection 5.3

Consider the number

$$x = \pm z.d_1d_2d_3 \dots d_{n-1}d_nd_{n+1} \dots,$$

where z is a (positive) integer and each d_i ($i = 1, 2, 3, \dots$) is a digit (i.e. an integer between 0 and 9 inclusive). The rounded form of the number x correct to n decimal places is

$$\bar{x} = \pm z.d_1d_2d_3 \dots d_{n-1}e_n,$$

where

$$e_n = \begin{cases} d_n, & \text{if } d_{n+1} = 0, 1, 2, 3, 4, \\ d_n + 1, & \text{if } d_{n+1} = 5, 6, 7, 8, 9. \end{cases}$$

We can similarly round to the nearest whole number, or 10, 100, 1000, etc.

Exercise 6

Round $-136.524\,863\,5$ to

- (i) two decimal places,
- (ii) four decimal places,
- (iii) six decimal places,
- (iv) the nearest whole number.

[Solution on page 70]

When rounding a number, as above, we inevitably make an approximation. The **error** in an approximation $\bar{x}$ to a number x is defined to be

$$e = \bar{x} - x.$$

Exercise 7

Round π to four decimal places. Find the error in this approximation, expressing your answer correct to six decimal places.

[Solution on page 70]

My calculator displays the value of π as $3.141\,592\,654$, which is correct to nine decimal places. This tells me that π has a value between $3.141\,592\,653\,5$ and $3.141\,592\,654\,5$. In other words, the error in my nine decimal place approximation for π is in the interval $[-0.000\,000\,000\,5, 0.000\,000\,000\,5]$. In general, the **error e_n in an approximation to n decimal places** satisfies

$$|e_n| \leq 5 \times 10^{-(n+1)}.$$

Instead of rounding a number to n decimal places, we can alternatively round a number to **n significant figures**. For example, 1234.56789 rounded to two significant figures is 1200 , to four significant figures is 1235 and to six significant figures is 1234.57 . Similarly, $0.000\,812\,376$ rounded to two and four significant figures is $0.000\,81$ and $0.000\,812\,4$ respectively.

Rounding to n significant figures means taking only n digits at the left-hand end of the decimal expression.

Exercise 8

Round the following numbers to four significant figures. In each case find the error in the approximation.

- (i) 324 961
- (ii) 0.001 743 498
- (iii) $-6.598\,374 \times 10^{-8}$

[Solution on page 70]

5.4 Fitting straight lines to data

In *Unit 7* we shall describe a simple experiment which you could set up at home. The experimental apparatus is constructed by tying together several rubber bands to produce an elastic string which is used to suspend a bag from a door lintel. The bag contains a number n of identical heavy coins, and the height h of the bag above the floor is measured when the system is stationary. The table below shows the experimental data for the height of the bag for different numbers of coins.

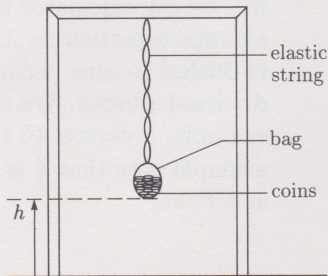


Figure 4 The experimental apparatus.

Table 1

Number n of coins	5	10	15	20	25
Height h above floor in metres	1.53	1.38	1.15	0.91	0.69

As you will see in the following exercise, a graphical plot of this data indicates that there is an approximately linear relationship between the height of the bag and the number of coins in the bag. In many experimental situations, including this one, it is sufficient to find the linear relationship by drawing a straight line ‘by eye’ which you think is the ‘best’ fit to the points in the graphical plot of the data values.

Exercise 9

- (i) On graph paper, plot the height, h , of the bag against the number, n , of coins in the bag for the data given in Table 1.
- (ii) Draw ‘by eye’ the straight line which you think is the ‘best’ fit to the points plotted in part (i).
- (iii) Use your answer to part (ii) to estimate the height of the bag before it contained any coins.
- (iv) Use your answer to part (ii) to estimate the decrease in the height of the bag caused by the addition of one coin to the bag.
- (v) Use your answers to parts (iii) and (iv) to write down the equation of your ‘best’ fit line.

[Solution on page 70]

If you wish to be more objective in finding the ‘best’ fit line, it is necessary to decide what you mean by ‘best’. The most common method is called **linear regression** or the **method of least squares**. The result is that the best linear fit

M101 Block II Unit 3
Section 3.4
MS284 Unit 7
Subsections 4.3 and 4.4

$$y = mx + c$$

to a set of data (x_i, y_i) ($i = 1, 2, 3, \dots, n$) has gradient

$$m = \frac{\frac{1}{n} \sum_{i=1}^n x_i y_i - \bar{x} \bar{y}}{\frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{x}^2}$$

and y -intercept

$$c = \bar{y} - m\bar{x},$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ and $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$.

The calculations are quite lengthy for a large number of data points, but you might find that your pocket calculator has the facility to perform the calculations automatically. Alternatively, you may have access to a computer software package (for example a spreadsheet package) which will perform the calculations. If you have to resort to hand calculations, you should work throughout to at least $k + 2$ significant figures, but quote your final answers to k significant figures, where the minimum accuracy given in the data is k significant figures.

Exercise 10

- (i) Either by performing hand calculations, using the linear regression facility on your pocket calculator, or by using this facility of a suitable software package, use the method of least squares to find the best linear fit to the data given in Table 1.
- (ii) Using the result of part (i), estimate the height of the bag above the floor when it contains 12 coins.

[Solution on page 70]

5.5 Taylor polynomials and series

Exercise 11

Find the slope of the tangent to the curve $y = e^x$ at $x = 0$. Hence find the equation of the tangent to the curve at the point $(0, 1)$.

[Solution on page 71]

In Exercise 11, you found the equation of the tangent to the curve $y = e^x$ at $x = 0$. This could be used to approximate the function $f(x) = e^x$ in the neighbourhood of $x = 0$, although it is not very good except for small values of x , as you can see from Figure 5(a). This approximation,

$$p_1(x) = 1 + x,$$

is found by demanding that it and its first derivative have the same values as $f(x) = e^x$ and its first derivative at $x = 0$. The above approximation can be improved by using a quadratic approximation whose second derivative also has the same value as that of $f(x)$ at $x = 0$.

Example 4

Find the quadratic approximation $p_2(x)$ to

$$f(x) = e^x$$

such that $p_2(x)$ and $f(x)$, their first derivatives and second derivatives have the same values at $x = 0$.

Solution

First we calculate the values of $f(x) = e^x$ and its derivatives at $x = 0$.

$$\begin{aligned} f(x) = e^x &\Rightarrow f(0) = 1 \\ f'(x) = e^x &\Rightarrow f'(0) = 1 \\ f''(x) = e^x &\Rightarrow f''(0) = 1 \end{aligned}$$

Next we calculate the derivatives of the quadratic function $p_2(x) = a_0 + a_1x + a_2x^2$ and their values at $x = 0$.

$$\begin{aligned} p_2(x) = a_0 + a_1x + a_2x^2 &\Rightarrow p_2(0) = a_0 \\ p'_2(x) = a_1 + 2a_2x &\Rightarrow p'_2(0) = a_1 \\ p''_2(x) = 2a_2 &\Rightarrow p''_2(0) = 2a_2 \end{aligned}$$

In order that $p_2(0) = f(0)$, $p'_2(0) = f'(0)$ and $p''_2(0) = f''(0)$, we must therefore choose

$$a_0 = 1, \quad a_1 = 1 \quad \text{and} \quad a_2 = \frac{1}{2}.$$

Hence the quadratic approximation to $f(x) = e^x$ is

$$p_2(x) = 1 + x + \frac{1}{2}x^2.$$

The graph of this function is shown in Figure 5(b). $\square$

Exercise 12

Consider again $f(x) = e^x$.

- (i) Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ and $f^{(4)}(0)$.
- (ii) Hence find the cubic approximation

$$p_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

such that $p_3(0) = f(0)$, $p'_3(0) = f'(0)$, $p''_3(0) = f''(0)$ and $p'''_3(0) = f'''(0)$.

- (iii) Further, find the quartic approximation

$$p_4(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

such that $p_4(x)$ and $f(x)$ and their first four derivatives have the same values at $x = 0$.

[Solution on page 71]

In the above exercises and example we have found the polynomial approximations

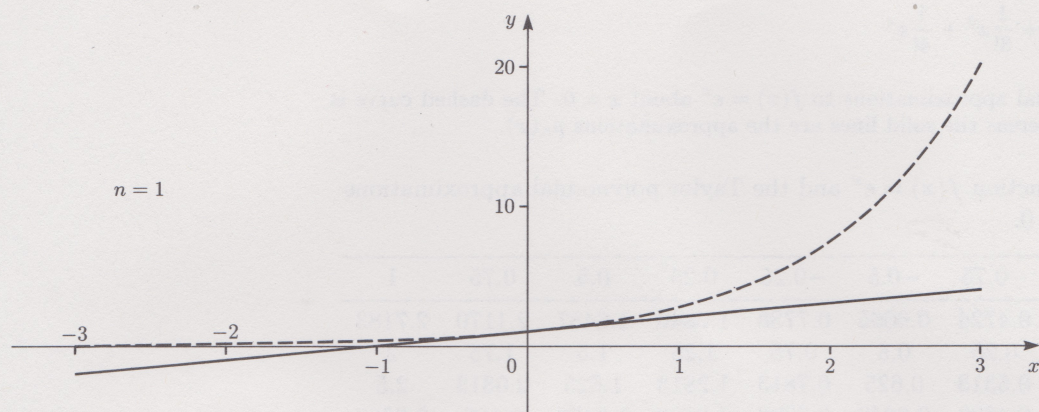
$$p_1(x) = 1 + \frac{1}{1!}x,$$

$$p_2(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2,$$

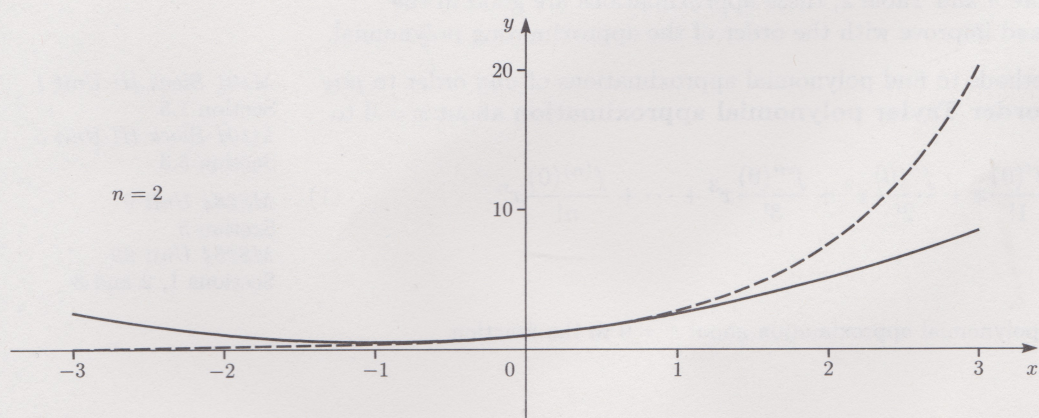
$$p_3(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3,$$

$$p_4(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

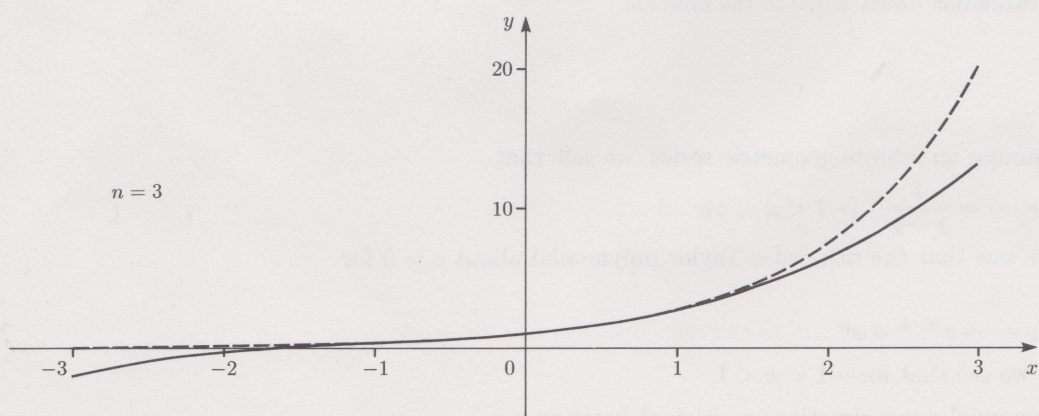
to the function $f(x) = e^x$ in the neighbourhood of $x = 0$. These are called the first-, second-, third- and fourth-order Taylor polynomial approximations about $x = 0$.



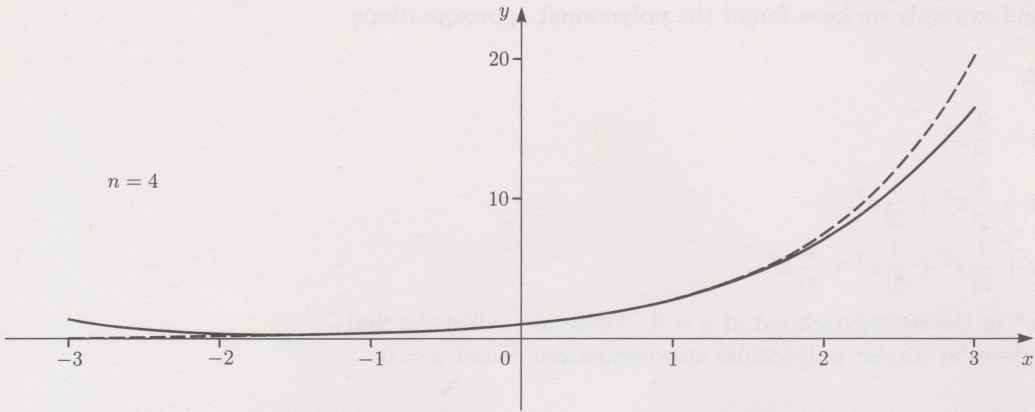
(a) $p_1(x) = 1 + x$



(b) $p_2(x) = 1 + x + \frac{1}{2!}x^2$



(c) $p_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$



(d) $p_4(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$

Figure 5 Taylor polynomial approximations to $f(x) = e^x$ about $x = 0$. The dashed curve is the function $f(x) = e^x$, whereas the solid lines are the approximations $p_n(x)$.

Table 2 The function $f(x) = e^x$ and the Taylor polynomial approximations $p_n(x)$ about $x = 0$.

x	-1	-0.75	-0.5	-0.25	0.25	0.5	0.75	1
$f(x)$	0.3679	0.4724	0.6065	0.7788	1.2840	1.6487	2.1170	2.7183
$p_1(x)$	0	0.25	0.5	0.75	1.25	1.5	1.75	2
$p_2(x)$	0.5	0.5313	0.625	0.7813	1.2813	1.625	2.0313	2.5
$p_3(x)$	0.3333	0.4609	0.6042	0.7786	1.2839	1.6458	2.102	2.6667
$p_4(x)$	0.375	0.4741	0.6068	0.7788	1.2840	1.6484	2.1147	2.7083

As you can see from Figure 5 and Table 2, these approximations are good in the neighbourhood of $x = 0$ and improve with the order of the approximating polynomial.

We can use the above methods to find polynomial approximations of *any* order to *any* function $f(x)$. The **n th-order Taylor polynomial approximation** about $x = 0$ to the function $f(x)$ is

$$p_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

(1)

Exercise 13

Find the n th-order Taylor polynomial approximation about $x = 0$ to the function

$$f(x) = \frac{1}{1-x}.$$

[Solution on page 71]

Supplementary Exercise 4

Find the cubic Taylor approximation about $x = 0$ to the function

$$f(x) = \tan x.$$

[Solution on page 79]

In Subsection 1.4, by summing an infinite geometric series, we saw that

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad (-1 < x < 1),$$

whereas in Exercise 13 we saw that the n th-order Taylor polynomial about $x = 0$ for $(1-x)^{-1}$ is

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n.$$

Comparing these results, we see that for $-1 < x < 1$

$$\lim_{n \rightarrow \infty} (\text{Taylor polynomial approximation}) = \text{original function}.$$

M101 Block III Unit 1
Section 1.5
M101 Block III Unit 5
Section 5.3
MS284 Unit 9
Section 5
MS284 Unit 99
Sections 1, 2 and 3

The question is whether this result is true for all functions. The answer is 'yes' as long as the function is 'well-behaved', perhaps for a restricted range for x . In the case of $f(x) = e^x$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

for all values of x . The infinite series obtained by taking the limit of a Taylor polynomial is called a **Taylor series**.

For a function $f(x)$, the Taylor series about $x = 0$ is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \quad (2)$$

We shall not discuss what 'well-behaved' means in this unit. You can assume that all functions you meet in this course are 'well-behaved'.

M101 Block III Unit 5
Section 5.3

MS284 Unit 99
Section 3

Exercise 14

Find the Taylor series about $x = 0$ for the function $f(x) = \sin x$.

[Solution on page 71]

The Taylor series about $x = 0$ for some functions are quoted in Subsection 6.4 of the *Handbook*. These are as follows.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (-1 < x < 1)$$

$$(a+x)^r = a^r + ra^{r-1}x + \frac{r(r-1)}{2!}a^{r-2}x^2 + \frac{r(r-1)(r-2)}{3!}a^{r-3}x^3 + \dots \quad (-a < x < a)$$

The last series is valid for any value of the constant r . If r is a positive integer, this reduces to the binomial expansion discussed in Subsection 1.2. Notice that the last two series in the table above are valid only for a restricted range of values of x .

Exercise 15

Use the above formulas to find the Taylor series about $x = 0$ for the following functions.

$$(i) f(x) = e^{-x} \quad (ii) f(x) = \log_e(1-2x) \quad \left(-\frac{1}{2} < x < \frac{1}{2}\right) \quad (iii) f(x) = x \sin x^2$$

[Solution on page 71]

Supplementary Exercise 5

Use the above formulas to find the first five terms of the Taylor series about $x = 0$ for the function

$$f(x) = (1+x)^{1/2}.$$

Hence evaluate $\sqrt{1.01}$ to 6 decimal places.

[Solution on page 80]

As illustrated in Figure 5 and Table 2, Taylor polynomials about $x = 0$ provide good approximations to the function for values of x near 0. However, we may wish to approximate a function in the neighbourhood of some other point. We can generalize expressions (1) and (2) to define the Taylor polynomials and Taylor series about $x = a$, where a may be any number. For example, the Taylor series for the function $f(x)$ about $x = a$ is

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

MS284 Unit 99
Section 4

Exercise 16

(i) Find the Taylor series about $x = 1$ for $f(x) = \log_e x$.

(ii) Use this series to calculate the value of $\log_e(1.1)$ to 4 decimal places.

[Solution on page 72]

6 End of unit exercises

Exercise 1

Find all the real values of x which satisfy the equation

$$x^4 = 2x^2 + 8.$$

Exercise 2

- (i) Show that ${}^nC_{n-r} = {}^nC_r$.
- (ii) What feature of Pascal's triangle does this formula describe?
- (iii) Use the formula in part (i) to evaluate ${}^{100}C_{98}$.

Exercise 3

Find expressions for $\sin \theta$, $\cos \theta$ and $\tan \theta$ in terms of $t = \tan \frac{1}{2}\theta$.

Exercise 4

Find all the angles θ in the range $0 \leq \theta < 2\pi$ which satisfy the equation

$$\sec^2 \theta = 4 \tan \theta + 6.$$

Exercise 5

Express $y = 2^x$ in the form $y = \exp(\text{something})$, and hence find its derivative.

Exercise 6

- (i) Write $\frac{A}{x+1} + \frac{B}{x+2}$, where A and B are constants, in the form $\frac{\text{expression}}{(x+1)(x+2)}$.
- (ii) Find values of A and B such that

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

- (iii) Hence evaluate

$$\int \frac{1}{(x+1)(x+2)} dx \quad (x > -1).$$

Exercise 7

Differentiate the following functions.

- (i) $\sin(x+x^2)$
- (ii) $\sin x + \sin^2 x$
- (iii) $\sin(\sin x)$
- (iv) $\sin(\sin(\sin x))$
- (v) $\frac{1+x}{1+x^2}$
- (vi) $(1+x^2)e^{-x} \sin 3x$

Exercise 8

Evaluate the following integrals.

- (i) $\int \tan^2 x dx$
- (ii) $\int \frac{x}{1+x} dx \quad (x > -1)$
- (iii) $\int \frac{x^2}{1+x^2} dx$
- (iv) $\int \frac{e^x}{1+e^x} dx$
- (v) $\int \frac{1}{x \log_e x} dx$
- (vi) $\int \frac{\log_e(\log_e x)}{x \log_e x} dx$
- (vii) $\int \frac{1}{\sqrt{1+e^x}} dx$

Exercise 9

A certain function $x(t)$ is known to take the form

$$x(t) = (A \cos 2t + B \sin t)e^t,$$

where A and B are constants. If $x(0) = 2$ and $\dot{x}(0) = 1$, find the values of A and B .

Exercise 10

Consider the equation

$$\frac{1}{2} \log_e \frac{v^2 - 1}{v^2 + 1} = gt + c.$$

- (i) Find the value of the constant c if $v = 2$ when $t = 0$.
- (ii) For the value of the constant c found in part (i), re-express the equation in a form whose subject is v .

Exercise 11

Find the values of the constants A and ϕ (where $A > 0$ and $0 \leq \phi < 2\pi$) such that

$$\cos \theta + 2 \sin \theta = A \cos(\theta + \phi)$$

for all θ .

Exercise 12

The passage of light through a lens (as in one eye-piece of a pair of spectacles) is governed by the formula

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v},$$

where u is the distance of the object from the lens, v is the distance from the lens of the image it forms (as illustrated in Figure 1) and f is a positive constant characteristic of the lens (called its focal length).

Find the position of the object for which the distance between the object and its image is a minimum. Further, find this minimum distance in terms of the constant f .

[Solutions on page 72]

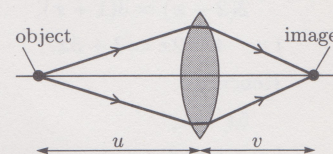


Figure 1

Appendix 1: Solutions to the exercises

Solutions to the exercises in Section 1

1. (i) $3(x + 3) - 7(x - 1) = 0.$

Multiplying out the brackets, we obtain

$$3x + 9 - 7x + 7 = 0.$$

Collecting together like terms gives

$$-4x + 16 = 0.$$

Putting the x -term and the constant on opposite sides of the equation leads to

$$4x = 16,$$

i.e. $x = \frac{16}{4} = 4.$

(ii) $\frac{2}{1+x} = \frac{3}{2-x}.$

Using the first method of Example 1, we obtain

$$2(2-x) = 3(1+x),$$

i.e. $4 - 2x = 3 + 3x.$

Hence

$$5x = 1,$$

i.e. $x = \frac{1}{5}.$

Alternatively, we collect together all the (non-zero) terms on the left-hand side of the equation to give

$$\frac{2}{1+x} - \frac{3}{2-x} = 0.$$

Putting the fractions over a common denominator, we obtain

$$\frac{2(2-x) - 3(1+x)}{(1+x)(2-x)} = 0.$$

Hence

$$\frac{4 - 2x - 3 - 3x}{(1+x)(2-x)} = 0,$$

i.e.

$$\frac{1 - 5x}{(1+x)(2-x)} = 0.$$

So $1 - 5x = 0$, i.e. $x = \frac{1}{5}$, as before.

2. (i) Rearranging the equation so that the term involving t is on one side of the equation and the other terms are on the other side, we have

$$\frac{1}{2}gt^2 = x_0 - x.$$

So

$$t^2 = \frac{2(x_0 - x)}{g},$$

hence

$$t = \pm \sqrt{\frac{2(x_0 - x)}{g}}.$$

Notice that

- in taking the square root, we have assumed that $2(x_0 - x)/g$ is positive;
- for each value of x , there are two possible values of t because of the $\pm$ sign, except for $x = x_0$ for which there is only one, namely $t = 0$.

(ii) We first square both sides of the equation, to give

$$\frac{x-2}{x+3} = t^2.$$

We now rearrange the equation so that the terms involving x are on one side of the equation and the other terms are on the other side:

$$\begin{aligned} x - 2 &= t^2(x + 3) \\ &= t^2x + 3t^2. \end{aligned}$$

Hence

$$x - t^2x = 2 + 3t^2,$$

i.e.

$$(1 - t^2)x = 2 + 3t^2,$$

so

$$x = \frac{2 + 3t^2}{1 - t^2}.$$

3. There are various ways of solving the equations; we shall give only one. If you have used a different method and obtained the same answer, then your working is almost certainly correct. The equations are

$$2x - y = 3,$$

$$3x + y = 2.$$

Adding the two equations, we obtain $5x = 5$, so that $x = 1$.

Substituting this value of x in the first equation gives $2 - y = 3$, so that $y = -1$. Hence the solution is $x = 1$, $y = -1$.

Finally, you should check that this answer satisfies the original equations.

4. (i) $x^2 - 5x + 4 = (x - 1)(x - 4)$

(ii) $4t^2 - 5t - 6 = (t - 2)(4t + 3)$

(iii) $4z^2 - 25 = (2z - 5)(2z + 5)$

(iv) $x^2 - 2xy - 24y^2 = (x - 6y)(x + 4y)$

(v) $\lambda^3 - 3\lambda^2 - 4\lambda = \lambda(\lambda^2 - 3\lambda - 4) = \lambda(\lambda - 4)(\lambda + 1)$

(vi) $u^2 - 6u + 9 = (u - 3)(u - 3) = (u - 3)^2$

5. (i) $x^2 - 5x + 6 = 0.$

Factorizing the expression gives

$$(x - 2)(x - 3) = 0.$$

Hence either $x - 2 = 0$ or $x - 3 = 0$. So

$$x = 2 \quad \text{or} \quad x = 3.$$

(ii) Rearranging the equation gives

$$p^2 - 10p + 25 = 0,$$

i.e. $(p - 5)^2 = 0.$

So the only solution is

$$p = 5.$$

(iii) $3y^2 + 8y + 4 = 0$ gives

$$(y + 2)(3y + 2) = 0.$$

So

$$y = -2 \quad \text{or} \quad y = -\frac{2}{3}.$$

$$6. (i) \quad x = \frac{1 \pm \sqrt{1+48}}{2} = \frac{1 \pm \sqrt{49}}{2} = \frac{1 \pm 7}{2} = 4 \text{ or } -3$$

$$(ii) \quad y = \frac{-4 \pm \sqrt{16-16}}{8} = \frac{-4}{8} = -\frac{1}{2}$$

(iii) As $b^2 - 4ac = 1 - 8 = -7$, which is negative, the equation has no real solutions.

$$(iv) \quad \lambda = \frac{-5 \pm \sqrt{25+12}}{2} \\ = \frac{-5 \pm \sqrt{37}}{2} \\ \simeq 0.5414 \text{ or } -5.541$$

$$7. (i) \quad x^2 - 4x - 5 = (x^2 - 4x) - 5 \\ = \{(x-2)^2 - 4\} - 5 \\ = (x-2)^2 - 9$$

(ii) In completed-square form, the equation $x^2 - 4x - 5 = 0$ becomes

$$(x-2)^2 - 9 = 0,$$

or

$$(x-2)^2 = 9.$$

Hence

$$x - 2 = \pm 3,$$

so

$$x = \pm 3 + 2 = 5 \text{ or } -1.$$

(iii) $x^2 - 4x - 5 = (x-2)^2 - 9$ from above.

As $(x-2)^2$ is non-negative, the quadratic expression has its minimum value when $x = 2$, and this minimum value is -9 .

$$8. (i) \quad (a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 \\ + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

(ii) The required row of Pascal's triangle is the next one after those shown on page 9, which is

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1.$$

So

$$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 \\ + 28a^2b^6 + 8ab^7 + b^8.$$

9. (i) Putting $a = x$, $b = -1$ and using the appropriate row of Pascal's triangle gives

$$(x-1)^5 = x^5 + 5 \times x^4 \times (-1) + 10 \times x^3 \times (-1)^2 \\ + 10 \times x^2 \times (-1)^3 + 5 \times x \times (-1)^4 + (-1)^5 \\ = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1.$$

(ii) Here we have $a = 1$, $b = -5x$ and $n = 4$, so

$$(1-5x)^4 = 1 + 4 \times (-5x) + 6 \times (-5x)^2 \\ + 4 \times (-5x)^3 + (-5x)^4 \\ = 1 - 20x + 150x^2 - 500x^3 + 625x^4.$$

(iii) Here we have $a = 3$, $b = 4x$ and $n = 3$, so

$$(3+4x)^3 = 3^3 + 3 \times 3^2 \times (4x) + 3 \times 3 \times (4x)^2 + (4x)^3 \\ = 27 + 108x + 144x^2 + 64x^3.$$

$$10. (i) \quad 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$(ii) \quad 7! = 7 \times 6 \times 5! = 5040$$

$$(iii) \quad {}^6C_3 = \frac{6!}{3!(6-3)!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (3 \times 2 \times 1)} \\ = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 5 \times 4 = 20$$

$$(iv) \quad {}^{20}C_2 = \frac{20!}{2!(20-2)!} \\ = \frac{20 \times 19 \times 18 \times 17 \times 16 \times \cdots \times 3 \times 2 \times 1}{(2 \times 1) \times (18 \times 17 \times 16 \times \cdots \times 3 \times 2 \times 1)} \\ = \frac{20 \times 19}{2 \times 1} = 190$$

11. (i) We have

$$\frac{n!}{(n-r)!} = \frac{n \times \cdots \times (n-r) \times \cdots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \cdots \times 3 \times 2 \times 1} \\ = n \times (n-1) \times \cdots \times (n-r+1),$$

so

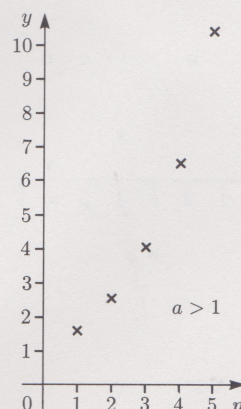
$${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n \times \cdots \times (n-r+1)}{r \times (r-1) \times \cdots \times 2 \times 1}.$$

$$(ii) \quad {}^nC_1 = \frac{n}{1} = n$$

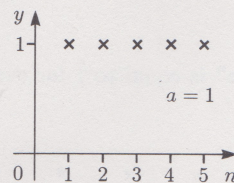
$${}^nC_2 = \frac{n \times (n-1)}{2 \times 1} = \frac{1}{2}n(n-1)$$

$$(iii) \quad {}^{20}C_3 = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 20 \times 19 \times 3 = 1140$$

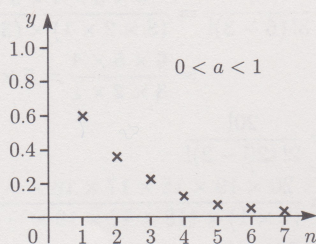
12. (i) For $a > 1$, a^n is a positive, rapidly increasing function of n .



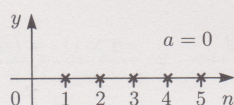
(ii) For $a = 1$, $a^n = 1$ for all n .



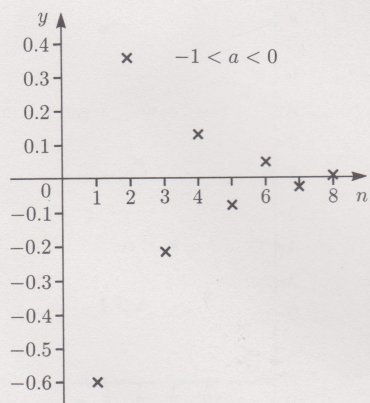
(iii) For $0 < a < 1$, a^n is a positive, decreasing function of n which tends to zero for large n .



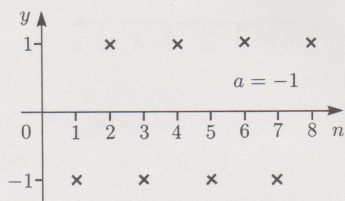
(iv) For $a = 0$, a^n is always equal to zero.



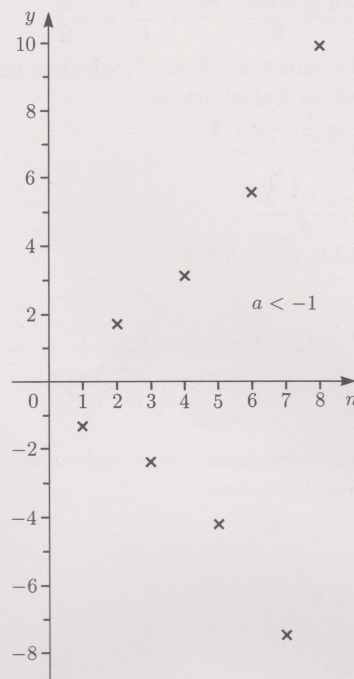
(v) For $-1 < a < 0$, a^n alternates in sign but its magnitude tends to zero for large n .



(vi) For $a = -1$, a^n is equal to 1 for even n and -1 for odd n .



(vii) For $a < -1$, a^n alternates in sign but its magnitude gets larger and larger with increasing n .



13. (i) $\sum_{r=1}^{10} r$

(ii) $\sum_{r=1}^N r^3$ (Note that $1^3 = 1$.)

(iii) $\sum_{r=1}^{\infty} \frac{1}{r!}$

(iv) $\sum_{r=0}^{\infty} \frac{x^r}{r!}$

(v) $\sum_{r=1}^{\infty} \frac{(-1)^{r+1} x^r}{r}$

Note that we could have used any symbol instead of r in all these answers, apart from the symbols N in part (ii) and x in parts (iv) and (v). Also note the use of $(-1)^r$ in part (v) for a series whose terms alternate in sign.

14. Initially there is 1 person in the chain, its initiator.

After the first posting, there are an additional 5 people in the chain.

After the second set of postings, there are an additional $5 \times 5 = 5^2$ people in the chain.

After the third set of postings, there are an additional $5 \times 5^2 = 5^3$ people in the chain.

So after the n th set of postings, there are an additional 5^n people in the chain. Hence after 10 sets of postings, the total number of people in the chain is

$$1 + 5 + 5^2 + 5^3 + \dots + 5^{10} = \frac{1(1 - 5^{11})}{1 - 5} = 12\,207\,031,$$

where we have used the formula for the sum of a geometric series with first term $a = 1$, common ratio $x = 5$ and number of terms $n = 11$.

15. The first term is $a = 1$ and the common ratio is $x = \frac{1}{2}$. So the sum of the infinite geometric series is

$$\frac{a}{1-x} = \frac{1}{1-\frac{1}{2}} = 2,$$

$$\text{i.e. } 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = 2.$$

$$\begin{aligned} 16. \quad 0.1111\ldots &= \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \left(\frac{1}{10}\right)^4 + \cdots \\ &= \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{9} \end{aligned}$$

Solutions to the exercises in Section 2

1. (i) (a) $\frac{1}{6}\pi$ (b) $\frac{1}{2}\pi$ (c) $\frac{3}{4}\pi$
 (d) $\frac{4}{3}\pi$ (e) 2π
 (ii) (a) 45° (b) 60° (c) 120°
 (d) 180° (e) 270°

2. (i) $\sec \frac{1}{6}\pi = \frac{1}{\cos \frac{1}{6}\pi} = \frac{2}{\sqrt{3}}$
 (ii) $\operatorname{cosec} \left(-\frac{1}{4}\pi\right) = \frac{1}{\sin\left(-\frac{1}{4}\pi\right)} = \frac{1}{-\sin \frac{1}{4}\pi} = -\sqrt{2}$
 (iii) $\cos \frac{5}{4}\pi = -\cos \frac{1}{4}\pi = -\frac{1}{\sqrt{2}}$
 (iv) $\tan \left(-\frac{11}{6}\pi\right) = \tan \frac{1}{6}\pi = \frac{1}{\sqrt{3}}$

3. $\theta = \frac{1}{6}\pi$ or $\frac{5}{6}\pi$.

This is illustrated in Figure 1.

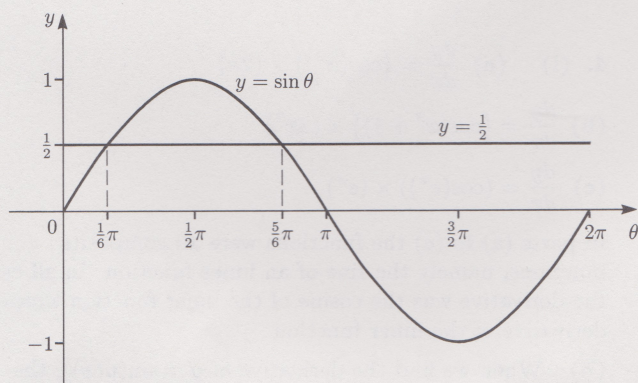


Figure 1

4. (i) $\arccos \frac{1}{2} = \frac{1}{3}\pi$ (ii) $\arctan 1 = \frac{1}{4}\pi$
 (iii) $\arccos\left(-\frac{1}{2}\right) = \frac{2}{3}\pi$ (iv) $\arctan(-1) = -\frac{1}{4}\pi$

5. We know $\cos^2 \theta = 1 - \sin^2 \theta = 1 - x^2$, so
 $\cos \theta = \pm \sqrt{1 - x^2}$.

But since θ is an acute angle, we conclude that

$$\cos \theta = +\sqrt{1 - x^2},$$

so

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{x}{\sqrt{1 - x^2}}.$$

We can, alternatively, derive these results by using the right-angled triangle in Figure 2.

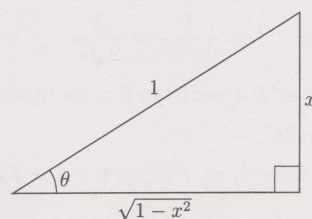


Figure 2

6. We know $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$, so
 $\sin\left(\frac{1}{3}\pi - \frac{1}{4}\pi\right) = \sin \frac{1}{3}\pi \cos \frac{1}{4}\pi - \cos \frac{1}{3}\pi \sin \frac{1}{4}\pi$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}},$$

$$\text{i.e. } \sin \frac{1}{12}\pi = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

7. We know $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, so
 $\sin(\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$,

$$\text{i.e. } \sin 2\alpha = 2 \sin \alpha \cos \alpha.$$

Similarly, $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$, so

$$\cos(\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha,$$

$$\text{i.e. } \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha. \quad (1)$$

Using the identity $\sin^2 \alpha = 1 - \cos^2 \alpha$ in Equation (1), we obtain

$$\begin{aligned} \cos 2\alpha &= \cos^2 \alpha - (1 - \cos^2 \alpha) \\ &= 2 \cos^2 \alpha - 1. \end{aligned}$$

Similarly, substituting $\cos^2 \alpha = 1 - \sin^2 \alpha$ in Equation (1) leads to

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha.$$

8. We have $\cos 2\alpha = 1 - 2 \sin^2 \alpha$, i.e.

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha).$$

Hence

$$\begin{aligned} \sin^2 \frac{1}{8}\pi &= \frac{1}{2}(1 - \cos \frac{1}{4}\pi) \\ &= \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}}\right) \\ &= \frac{\sqrt{2}-1}{2\sqrt{2}}. \end{aligned}$$

As $\sin \frac{1}{8}\pi$ is positive, we finally obtain

$$\sin \frac{1}{8}\pi = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}.$$

9. Putting $\beta = 2\alpha$ in the formula for $\sin(\alpha + \beta)$, we obtain
 $\sin 3\alpha = \sin \alpha \cos 2\alpha + \cos \alpha \sin 2\alpha$.

But $\cos 2\alpha = 1 - 2 \sin^2 \alpha$ and $\sin 2\alpha = 2 \sin \alpha \cos \alpha$, so

$$\begin{aligned} \sin 3\alpha &= (\sin \alpha) \times (1 - 2 \sin^2 \alpha) + (\cos \alpha) \times (2 \sin \alpha \cos \alpha) \\ &= \sin \alpha - 2 \sin^3 \alpha + 2 \sin \alpha \cos^2 \alpha \\ &= \sin \alpha - 2 \sin^3 \alpha + 2 \sin \alpha (1 - \sin^2 \alpha) \\ &= 3 \sin \alpha - 4 \sin^3 \alpha. \end{aligned}$$

Solutions to the exercises in Section 3

1. (i) $f'(x) = 2 \cos x - \sin x$

(ii) $\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

(iii) $v'(\theta) = \sec^2 \theta + \sec \theta \tan \theta = \sec \theta (\sec \theta + \tan \theta)$

(iv) $\dot{x} = 2t + 2e^t$

(v) $f'(x) = 6x - 4$, so $f'(3) = 6 \times 3 - 4 = 14$.

(vi) $\frac{dy}{dt} = 3 \sec^2 t - 2 \cos t$,

so $\frac{dy}{dt}(0) = 3 \sec^2 0 - 2 \cos 0 = 3 - 2 = 1$.

(vii) $\frac{dz}{dx} = \frac{1}{x} = x^{-1}$, so $\frac{d^2z}{dx^2} = -x^{-2} = -\frac{1}{x^2}$.

(viii) $f'(t) = 2 \cos t$, $f''(t) = -2 \sin t$, $f'''(t) = -2 \cos t$,
 $f^{(4)}(t) = 2 \sin t$, $f^{(5)}(t) = 2 \cos t$, $f^{(6)}(t) = -2 \sin t$.

So $f^{(6)}(\frac{1}{4}\pi) = -2 \sin(\frac{1}{4}\pi) = -2/\sqrt{2} = -\sqrt{2}$.

2. There are two different methods of solution, depending on whether you use the function or Leibniz notation. In the solutions below, we shall alternate the approaches. However, we advise you to stick to one method, choosing the one which is most familiar to you.

(i) Here $y = \sin u$, where $u = 2x + \frac{1}{5}\pi$.

So $\frac{dy}{du} = \cos u = \cos(2x + \frac{1}{5}\pi)$ and $\frac{du}{dx} = 2$.

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 2 \cos(2x + \frac{1}{5}\pi).$$

(ii) Here $s = f(g(t))$, where $f(u) = e^u$ and $g(t) = 5t$.

So $f'(u) = e^u$ and $g'(t) = 5$.

Hence

$$\frac{ds}{dt} = f'(g(t))g'(t) = (e^{5t}) \times (5) = 5e^{5t}.$$

(iii) Here $y = \arcsin u$, where $u = 5x$.

So $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} = \frac{1}{\sqrt{1-25x^2}}$ and $\frac{du}{dx} = 5$.

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{5}{\sqrt{1-25x^2}}.$$

(iv) Here $z = f(g(x))$, where $f(y) = y^3$ and $g(x) = 3x^2 + 1$.

So $f'(y) = 3y^2$ and $g'(x) = 6x$.

Hence

$$\begin{aligned} \frac{dz}{dx} &= f'(g(x))g'(x) \\ &= \{3(3x^2 + 1)^2\} \times \{6x\} \\ &= 18x(3x^2 + 1)^2. \end{aligned}$$

(v) Here $z = \log_e u$, where $u = y^3 + 1$.

So $\frac{dz}{du} = \frac{1}{u} = \frac{1}{y^3 + 1}$ and $\frac{du}{dy} = 3y^2$.

Hence

$$\frac{dz}{dy} = \frac{dz}{du} \frac{du}{dy} = \frac{3y^2}{y^3 + 1}.$$

(vi) Here $y = f(g(x))$, where $f(z) = z^{1/2}$ and $g(x) = 4 - x^2$.

So $f'(z) = \frac{1}{2}z^{-1/2}$ and $g'(x) = -2x$.

Hence

$$\begin{aligned} \frac{dy}{dx} &= f'(g(x))g'(x) \\ &= \left\{\frac{1}{2}(4 - x^2)^{-1/2}\right\} \times \{-2x\} \\ &= -\frac{x}{\sqrt{4 - x^2}}. \end{aligned}$$

3. (i) Using the product rule,

$$\frac{dy}{dt} = (1)(\sin t) + (t)(\cos t) = \sin t + t \cos t.$$

(ii) Using the quotient rule,

$$\begin{aligned} \frac{du}{d\theta} &= \frac{(\cos \theta)(\cos \theta) - (\sin \theta)(-\sin \theta)}{(\cos \theta)^2} \\ &= \frac{1}{\cos^2 \theta} = \sec^2 \theta, \end{aligned}$$

using the trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$.

[Note: As $\sin \theta / \cos \theta = \tan \theta$, we could have obtained this result by using the table of standard derivatives.]

(iii) Using the chain rule and the quotient rule,

$$\begin{aligned} \frac{dy}{dx} &= \sec\left(\frac{x}{x^2+1}\right) \tan\left(\frac{x}{x^2+1}\right) \times \frac{(1)(x^2+1) - (x)(2x)}{(x^2+1)^2} \\ &= \frac{1-x^2}{(x^2+1)^2} \sec\left(\frac{x}{x^2+1}\right) \tan\left(\frac{x}{x^2+1}\right). \end{aligned}$$

(iv) Using the chain rule and the product rule,

$$\begin{aligned} \frac{dx}{dt} &= \cos(t \sin t) \times \{(1)(\sin t) + (t)(\cos t)\} \\ &= (\sin t + t \cos t) \cos(t \sin t). \end{aligned}$$

(v) Using the product rule and the chain rule,

$$\begin{aligned} \frac{dx}{dy} &= (\cos y e^{\sin y})(\tan y) + (e^{\sin y})(\sec^2 y) \\ &= (\sin y + \sec^2 y)e^{\sin y}, \end{aligned}$$

using $\tan y = \sin y / \cos y$.

4. (i) (a) $\frac{dy}{dx} = (\cos(x^2)) \times (2x)$

(b) $\frac{dy}{dx} = (\cos(x^3 + 1)) \times (3x^2)$

(c) $\frac{dy}{dx} = (\cos(e^x)) \times (e^x)$

In parts (a) to (c) the functions were all composite functions, namely the sine of an inner function. In all cases the derivative was the cosine of the inner function times the derivative of the inner function.

(ii) When we find the derivative of $y = \sin(u(x))$, the answer will also satisfy the above pattern. So

$$\frac{dy}{dx} = \cos(u(x)) \times \frac{du}{dx}.$$

5. (i) Using the chain rule for differentiating a 'function of a function',

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \frac{dy}{dx} = 2y \frac{dy}{dx}.$$

(ii) $\frac{d}{dx}(\tan y) = \frac{d}{dy}(\tan y) \frac{dy}{dx} = \sec^2 y \frac{dy}{dx}$

(iii) $\frac{d}{dx}(\log_e y) = \frac{d}{dy}(\log_e y) \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx}$

(iv) $\begin{aligned} \frac{d}{dx}(\log_e(\sin y)) &= \frac{d}{dy}(\log_e(\sin y)) \frac{dy}{dx} \\ &= \frac{\cos y}{\sin y} \frac{dy}{dx} = \cot y \frac{dy}{dx} \end{aligned}$

(v) Here we have to use the product rule:

$$\frac{d}{dx}(x^2y) = y \frac{d}{dx}(x^2) + x^2 \frac{dy}{dx} = 2xy + x^2 \frac{dy}{dx}.$$

$$\begin{aligned} \text{(vi)} \quad \frac{d}{dx}(y^2 e^x) &= y^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(y^2) \\ &= y^2 e^x + e^x \frac{d}{dy}(y^2) \frac{dy}{dx} \\ &= y^2 e^x + 2ye^x \frac{dy}{dx} \end{aligned}$$

$$\begin{aligned} \text{(vii)} \quad \frac{d}{dx}(\sin(x^2 + y^2)) &= \cos(x^2 + y^2) \frac{d}{dx}(x^2 + y^2) \\ &= \cos(x^2 + y^2) \left(2x + 2y \frac{dy}{dx} \right) \\ &= 2 \left(x + y \frac{dy}{dx} \right) \cos(x^2 + y^2) \end{aligned}$$

6. (i) Solving the equation of the ellipse for y in terms of x gives

$$y = \pm \frac{1}{2} \sqrt{8 - x^2}.$$

The two signs correspond to the curve in the upper and lower half-planes, there being two possible values of y for any value of x in the range $-\sqrt{8} < x < \sqrt{8}$. We are interested in the point $(2, 1)$, which is in the upper half-plane and so is on the curve

$$y = \frac{1}{2} \sqrt{8 - x^2}.$$

Hence $\frac{dy}{dx} = -\frac{x}{2\sqrt{8 - x^2}}.$

So the slope of the tangent to the curve at the point $(2, 1)$ is

$$\frac{dy}{dx}(2) = -\frac{2}{2\sqrt{8 - 4}} = -\frac{1}{2}.$$

(ii) Differentiating $x^2 + 4y^2 = 8$ implicitly gives

$$2x + 8y \frac{dy}{dx} = 0,$$

i.e. $\frac{dy}{dx} = -\frac{x}{4y}.$

So at the point $(2, 1)$,

$$\frac{dy}{dx} = -\frac{2}{4} = -\frac{1}{2},$$

which agrees with the answer obtained in part (i).

7. Using implicit differentiation,

$$3x^2 + 2xy + x^2 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0.$$

Hence $\frac{dy}{dx} = -\frac{3x^2 + 2xy}{x^2 + 3y^2}.$

So at the point $(-1, 1)$, the slope of the tangent to the curve is

$$\frac{dy}{dx} = -\frac{3 - 2}{1 + 3} = -\frac{1}{4}.$$

8. In this solution we shall use Leibniz notation, rather than the function notation used in the solution to Example 2.

$$y = 2x^3 - 3x^2 - 12x + 6,$$

so $\frac{dy}{dx} = 6x^2 - 6x - 12$ and $\frac{d^2y}{dx^2} = 12x - 6.$

At stationary points, $\frac{dy}{dx} = 0$, so

$$6x^2 - 6x - 12 = 0,$$

i.e. $x^2 - x - 2 = 0$, or $(x + 1)(x - 2) = 0.$

Thus $x = -1$ or $x = 2.$

When $x = -1$,

$$\frac{d^2y}{dx^2} = 12 \times (-1) - 6 = \text{negative}$$

and $y = 2 \times (-1)^3 - 3 \times (-1)^2 - 12 \times (-1) + 6 = 13.$

So the stationary point $(-1, 13)$ is a local maximum.

When $x = 2$,

$$\frac{d^2y}{dx^2} = 12 \times 2 - 6 = \text{positive}$$

and $y = 2 \times 2^3 - 3 \times 2^2 - 12 \times 2 + 6 = -14.$

Therefore the stationary point $(2, -14)$ is a local minimum.

9. (i)

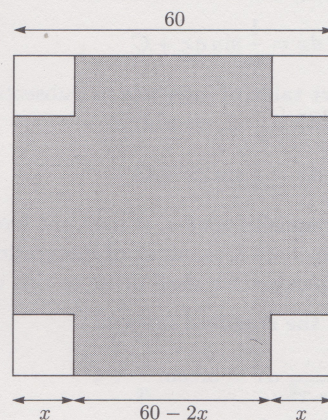


Figure 1

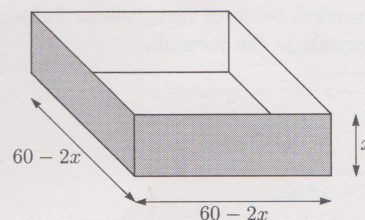


Figure 2

As shown in Figures 1 and 2, the volume of the box in cm^3 is

$$\begin{aligned} v &= x(60 - 2x)^2 \\ &= 3600x - 240x^2 + 4x^3. \end{aligned}$$

Note that for physical reasons, we need to restrict the domain of this function to $0 \leq x \leq 30$.

(ii) Now $\frac{dv}{dx} = 3600 - 480x + 12x^2 = 12(10 - x)(30 - x)$

and $\frac{d^2v}{dx^2} = -480 + 24x.$

At stationary points, $\frac{dv}{dx} = 0$, which leads to $x = 10$ or $x = 30$.

When $x = 10$,

$$\frac{d^2v}{dx^2} = -480 + 24 \times 10 = \text{negative},$$

which means that this is a local maximum, whose value is

$$v = 10(60 - 2 \times 10)^2 = 16000.$$

When $x = 30$,

$$\frac{d^2v}{dx^2} = -480 + 24 \times 30 = \text{positive},$$

which means that this is a local minimum.

At the two end-points of the domain, $x = 0$ and $x = 30$, we have $v = 0$.

So the maximum volume occurs when $x = 10$ and its value is $16\,000\text{ cm}^3$.

Alternatively, we could argue that at the two end-points of the domain, $x = 0$ and $x = 30$, we have $v = 0$. Within the physical domain the volume v is positive. As there is only one stationary point in this region, namely at $x = 10$, this stationary point must be a maximum.

Solutions to the exercises in Section 4

1. (i) Using the standard integral

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C$$

from the first table of integrals in Subsection 7.1 of the *Handbook* with $a = 2$, we obtain

$$\int \cos 2x \, dx = \frac{1}{2} \sin 2x + C.$$

In your prerequisite course, it may not have been the practice to include a constant of integration C in indefinite integrals. However, we shall do so in this course.

(ii) Using the standard integral

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \frac{x}{a} + C$$

with $a = 3$ leads to

$$\int \frac{1}{\sqrt{9 - x^2}} \, dx = \arcsin \frac{x}{3} + C.$$

(iii) The second table of integrals in Subsection 7.1 of the *Handbook* contains the formula

$$\int \sqrt{a^2 - x^2} \, dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C.$$

Hence

$$\begin{aligned} \int_{-1}^1 \sqrt{4 - x^2} \, dx &= \left[2 \arcsin \frac{x}{2} + \frac{x}{2} \sqrt{4 - x^2} \right]_{-1}^1 \\ &= (2 \arcsin \frac{1}{2} + \frac{1}{2} \sqrt{3}) - (2 \arcsin(-\frac{1}{2}) - \frac{1}{2} \sqrt{3}) \\ &= (\frac{1}{3}\pi + \frac{1}{2}\sqrt{3}) - (-\frac{1}{3}\pi - \frac{1}{2}\sqrt{3}) \\ &= \frac{2}{3}\pi + \sqrt{3}. \end{aligned}$$

(iv) As $2z + 3 > 0$ for all z in the interval of integration $0 \leq z \leq 1$, we use the standard integral

$$\int \frac{1}{ax + b} \, dx = \frac{1}{a} \log_e(ax + b) + C \quad \text{for } ax + b > 0$$

with $a = 2$, $b = 3$, and the variable x replaced by the variable z , to obtain

$$\begin{aligned} \int_0^1 \frac{1}{2z + 3} \, dz &= \left[\frac{1}{2} \log_e(2z + 3) \right]_0^1 \\ &= \frac{1}{2} \log_e 5 - \frac{1}{2} \log_e 3 \\ &= \frac{1}{2} \log_e \frac{5}{3}. \end{aligned}$$

(v) Using the standard integral

$$\int \sin^2 ax \, dx = \frac{1}{2}x - \frac{1}{4a} \sin 2ax + C$$

from the third table of integrals gives

$$\int \sin^2 3\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{12} \sin 6\theta + C.$$

2. (i) My first guess is $\sin(2x + 3)$. However,

$$\frac{d}{dx}(\sin(2x + 3)) = 2 \cos(2x + 3),$$

so my corrected guess is $F(x) = \frac{1}{2} \sin(2x + 3)$. Finally, I check that

$$\frac{d}{dx}(\frac{1}{2} \sin(2x + 3)) = \cos(2x + 3).$$

$$\text{Hence } \int \cos(2x + 3) \, dx = \frac{1}{2} \sin(2x + 3) + C.$$

(ii) My first guess is e^{x^2} . However,

$$\frac{d}{dx}(e^{x^2}) = 2xe^{x^2},$$

so I adjust my guess to $F(x) = \frac{1}{2}e^{x^2}$. Finally, I check that

$$\frac{d}{dx}(\frac{1}{2}e^{x^2}) = xe^{x^2},$$

as required, so that I can conclude that

$$\int xe^{x^2} \, dx = \frac{1}{2}e^{x^2} + C.$$

(iii) My initial guess is $(x^3 + 1)^4$. However,

$$\frac{d}{dx}((x^3 + 1)^4) = 12x^2(x^3 + 1)^3,$$

so I adjust my guess to $F(x) = \frac{1}{12}(x^3 + 1)^4$. I check that

$$\frac{d}{dx}(\frac{1}{12}(x^3 + 1)^4) = x^2(x^3 + 1)^3,$$

which means that

$$\int x^2(x^3 + 1)^3 \, dx = \frac{1}{12}(x^3 + 1)^4 + C.$$

(iv) Concentrating on the function $(x^2 - 1)^{1/2}$ in the integrand and recalling the standard integral

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C,$$

my initial guess is $(x^2 - 1)^{3/2}$. However,

$$\frac{d}{dx}((x^2 - 1)^{3/2}) = 3x(x^2 - 1)^{1/2},$$

so I adjust my guess to $F(x) = \frac{1}{3}(x^2 - 1)^{3/2}$. Finally, I check that

$$\frac{d}{dx}(\frac{1}{3}(x^2 - 1)^{3/2}) = x(x^2 - 1)^{1/2},$$

and conclude that

$$\int x\sqrt{x^2 - 1} \, dx = \frac{1}{3}(x^2 - 1)^{3/2} + C.$$

3. (i) Let $u = x^2 - 5$, so that $\frac{du}{dx} = 2x$, or $du = 2x \, dx$. Hence

$$\begin{aligned} \int x(x^2 - 5)^7 \, dx &= \frac{1}{2} \int (x^2 - 5)^7 (2x \, dx) \\ &= \frac{1}{2} \int u^7 \, du \\ &= \frac{1}{16} u^8 + C \\ &= \frac{1}{16} (x^2 - 5)^8 + C. \end{aligned}$$

(ii) Let $u = 1 - t^2$, so that $\frac{du}{dt} = -2t$, or $du = -2t \, dt$. Hence

$$\begin{aligned} \int \frac{t}{1 - t^2} \, dt &= -\frac{1}{2} \int \left(\frac{1}{1 - t^2} \right) (-2t \, dt) \\ &= -\frac{1}{2} \int \frac{1}{u} \, du \\ &= -\frac{1}{2} \log_e u + C \quad (\text{since } u > 0) \\ &= -\frac{1}{2} \log_e(1 - t^2) + C. \end{aligned}$$

(iii) Let $u = 1 - z^4$, so that $\frac{du}{dz} = -4z^3$, or $du = -4z^3 dz$.
Hence

$$\begin{aligned}\int z^3(1 - z^4)^6 dz &= -\frac{1}{4} \int (1 - z^4)^6 (-4z^3 dz) \\ &= -\frac{1}{4} \int u^6 du \\ &= -\frac{1}{28} u^7 + C \\ &= -\frac{1}{28} (1 - z^4)^7 + C.\end{aligned}$$

(iv) Let $u = x + 1$, i.e. $x = u - 1$, so that $dx = du$.

When $x = 0$, $u = 1$, and when $x = 7$, $u = 8$. Hence

$$\begin{aligned}\int_{x=0}^{x=7} x(x+1)^{1/3} dx &= \int_{u=1}^{u=8} (u-1)u^{1/3} du \\ &= \int_1^8 (u^{4/3} - u^{1/3}) du \\ &= \left[\frac{3}{7} u^{7/3} - \frac{3}{4} u^{4/3} \right]_{u=1}^{u=8} \\ &= \left(\frac{3}{7} \times 8^{7/3} - \frac{3}{4} \times 8^{4/3} \right) \\ &\quad - \left(\frac{3}{7} \times 1^{7/3} - \frac{3}{4} \times 1^{4/3} \right) \\ &= \frac{3}{7} \times 128 - \frac{3}{4} \times 16 - \frac{3}{7} + \frac{3}{4} \\ &= 43 \frac{5}{28}.\end{aligned}$$

(v) We use the trigonometric substitution

$$x = 2 \tan u.$$

Hence $\frac{dx}{du} = 2 \sec^2 u$ and $dx = 2 \sec^2 u du$.

Thus

$$\begin{aligned}\int \frac{1}{(4+x^2)^2} dx &= \int \frac{1}{(4(1+\tan^2 u))^2} (2 \sec^2 u du) \\ &= \int \frac{2 \sec^2 u}{16 \sec^4 u} du \\ &= \frac{1}{8} \int \frac{1}{\sec^2 u} du \\ &= \frac{1}{8} \int \cos^2 u du \\ &= \frac{1}{16} \int (1 + \cos 2u) du \\ &= \frac{1}{16} u + \frac{1}{32} \sin 2u + C \\ &= \frac{1}{16} u + \frac{1}{16} \sin u \cos u + C \\ &= \frac{1}{16} \arctan\left(\frac{1}{2}x\right) \\ &\quad + \frac{1}{16} \sin(\arctan(\frac{1}{2}x)) \cos(\arctan(\frac{1}{2}x)) + C \\ &= \frac{1}{16} \arctan\left(\frac{1}{2}x\right) + \frac{1}{16} \frac{x}{\sqrt{4+x^2}} \frac{2}{\sqrt{4+x^2}} + C \\ &= \frac{1}{16} \arctan\left(\frac{1}{2}x\right) + \frac{x}{8(4+x^2)} + C,\end{aligned}$$

where we have used the right-angled triangle in Figure 1 to simplify $\sin(\arctan(\frac{1}{2}x))$ and $\cos(\arctan(\frac{1}{2}x))$.

In fact, this result is true for all x , not just positive x , as can be verified by differentiation.

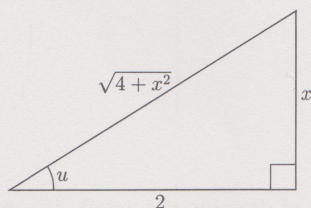


Figure 1

(vi) Let $u = x - 1$, which can be written as $x = u + 1$. So $dx = du$.

When $x = 1$, $u = 0$, and when $x = 2$, $u = 1$.

Hence

$$\begin{aligned}\int_{x=1}^{x=2} x\sqrt{x-1} dx &= \int_{u=0}^{u=1} (u+1)u^{1/2} du \\ &= \int_0^1 (u^{3/2} + u^{1/2}) du \\ &= \left[\frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right]_{u=0}^{u=1} \\ &= \frac{2}{5} + \frac{2}{3} = \frac{16}{15}.\end{aligned}$$

4. (i) Using the priority list given in the text, we choose

$$u = \log_e x \quad \text{and} \quad \frac{dv}{dx} = x.$$

So $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{1}{2}x^2$.

Substituting in the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

we have

$$\begin{aligned}\int x \log_e x dx &= (\log_e x) \left(\frac{1}{2}x^2 \right) - \int \left(\frac{1}{2}x^2 \right) \left(\frac{1}{x} \right) dx \\ &= \frac{1}{2}x^2 \log_e x - \frac{1}{2} \int x dx \\ &= \frac{1}{2}x^2 \log_e x - \frac{1}{4}x^2 + C \\ &= \frac{1}{4}x^2 (2 \log_e x - 1) + C.\end{aligned}$$

(ii) We choose

$$u = x^2 \quad \text{and} \quad \frac{dv}{dx} = e^x.$$

So $\frac{du}{dx} = 2x$ and $v = e^x$.

Substituting in the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

gives

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx. \quad (1)$$

To evaluate $\int x e^x dx$ we use integration by parts for a second time. This time we take

$$u = x \quad \text{and} \quad \frac{dv}{dx} = e^x.$$

In this case we have $\frac{du}{dx} = 1$ and $v = e^x$.

So integration by parts gives

$$\begin{aligned}\int x e^x dx &= x e^x - \int e^x dx \\ &= x e^x - e^x + C.\end{aligned}$$

Substituting this in Equation (1) gives

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - 2(x e^x - e^x + C) \\ &= (x^2 - 2x + 2) e^x + C',\end{aligned}$$

where $C' = -2C$.

(iii) Using the hint in the question, we take

$$u = \log_e x \quad \text{and} \quad \frac{dv}{dx} = 1.$$

So $\frac{du}{dx} = \frac{1}{x}$ and $v = x$.

The formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

leads to

$$\begin{aligned} \int \log_e x dx &= x \log_e x - \int (x) \left(\frac{1}{x} \right) dx \\ &= x \log_e x - \int 1 dx \\ &= x \log_e x - x + C \\ &= x (\log_e x - 1) + C. \end{aligned}$$

5. If we set

$$u = e^x \quad \text{and} \quad \frac{dv}{dx} = \sin x,$$

we have

$$\frac{du}{dx} = e^x \quad \text{and} \quad v = -\cos x.$$

So substituting in the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

we obtain

$$I = \int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx. \quad (2)$$

We now use integration by parts for a second time, for the integral $\int e^x \cos x dx$. We take

$$u = e^x \quad \text{and} \quad \frac{dv}{dx} = \cos x,$$

$$\text{so} \quad \frac{du}{dx} = e^x \quad \text{and} \quad v = \sin x.$$

Hence

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

Substituting this result in Equation (2) gives

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx + C,$$

$$\text{i.e.} \quad I = -e^x \cos x + e^x \sin x - I + C.$$

$$\text{Hence} \quad 2I = e^x (\sin x - \cos x) + C,$$

$$\text{i.e.} \quad I = \frac{1}{2} e^x (\sin x - \cos x) + C',$$

$$\text{where} \quad C' = \frac{1}{2} C.$$

6. In the integration by parts formula we take

$$u = x \quad \text{and} \quad \frac{dv}{dx} = \sin 2x.$$

$$\text{So} \quad \frac{du}{dx} = 1 \quad \text{and} \quad v = -\frac{1}{2} \cos 2x.$$

Substituting in the formula

$$\int_a^b u \frac{dv}{dx} dx = [uv]_{x=a}^{x=b} - \int_a^b v \frac{du}{dx} dx$$

gives

$$\begin{aligned} \int_0^{\pi/2} x \sin 2x dx &= \left[-\frac{1}{2} x \cos 2x \right]_{x=0}^{x=\pi/2} + \frac{1}{2} \int_0^{\pi/2} \cos 2x dx \\ &= -\frac{1}{4} \pi \cos \pi + \frac{1}{2} \left[\frac{1}{2} \sin 2x \right]_{x=0}^{x=\pi/2} \\ &= \frac{1}{4} \pi + \frac{1}{2} \left(\frac{1}{2} \sin \pi - \frac{1}{2} \sin 0 \right) \\ &= \frac{1}{4} \pi. \end{aligned}$$

$$\begin{aligned} 7. \text{ (i)} \quad I_0 &= \int_0^\infty e^{-x} dx \quad (\text{as } x^0 = 1) \\ &= \left[-e^{-x} \right]_{x=0}^{x=\infty} = 1. \end{aligned}$$

(ii) In the integration by parts formula we take

$$u = x^n \quad \text{and} \quad \frac{dv}{dx} = e^{-x}.$$

$$\text{Therefore} \quad \frac{du}{dx} = nx^{n-1} \quad \text{and} \quad v = -e^{-x}.$$

Substituting in the formula

$$\int_a^b u \frac{dv}{dx} dx = [uv]_{x=a}^{x=b} - \int_a^b v \frac{du}{dx} dx$$

leads to

$$\begin{aligned} \int_0^\infty x^n e^{-x} dx &= -[x^n e^{-x}]_{x=0}^{x=\infty} + n \int_0^\infty x^{n-1} e^{-x} dx \\ &= n \int_0^\infty x^{n-1} e^{-x} dx, \end{aligned}$$

$$\text{i.e.} \quad I_n = n I_{n-1}.$$

(iii) Using the result of part (ii) repeatedly, we have

$$\begin{aligned} I_n &= n I_{n-1} \\ &= n(n-1) I_{n-2} \\ &= n(n-1)(n-2) I_{n-3} \\ &\vdots \\ &= n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1 I_0 \\ &= n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1, \end{aligned}$$

as we showed that $I_0 = 1$ in part (i). Hence

$$I_n = n!$$

for $n = 1, 2, 3, \dots$ (As $0! = 1$, this result is also true for $n = 0$.)

Solutions to the exercises in Section 5

1. (i) $2x + 3y = 3$.

When $x = 0$ we have $3y = 3$, i.e. $y = 1$. So the y -intercept is 1. Similarly, when $y = 0$ we have $2x = 3$, i.e. $x = \frac{3}{2}$. Hence the x -intercept is $\frac{3}{2}$. Using these two intercepts we can draw the graph of the straight line. This is shown in Figure 1.

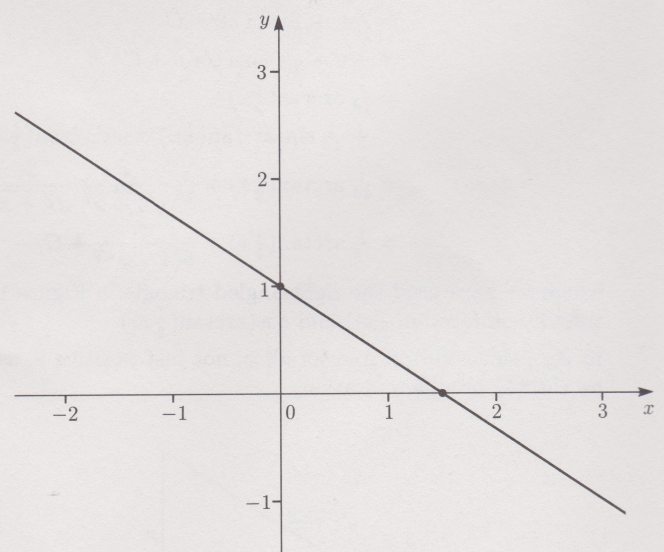


Figure 1

This method of drawing the graph of a straight line by finding the two intercepts is usually the most convenient.

(ii) $y = \frac{1}{2}x$.

This straight line obviously passes through the origin. So we can not use intercepts to draw the graph of the line. Instead, we need one further point on the line, and one such is $(2, 1)$. So the graph of the straight line is as shown in Figure 2.

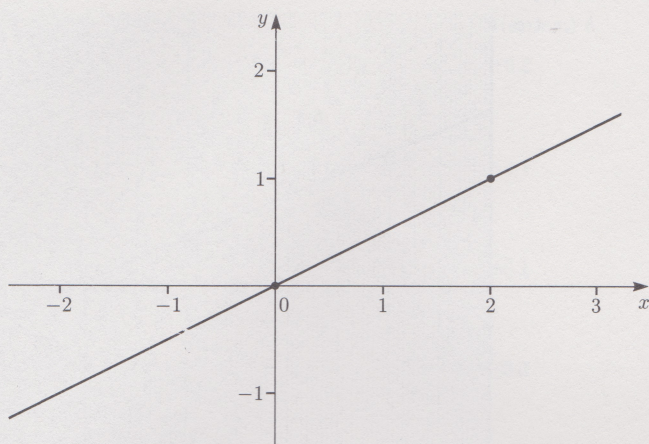


Figure 2

2. We write the equation of the straight line as

$$y = mx + c.$$

We are told that the slope is -2 , so $m = -2$.

Also, the point $(1, 1)$ lies on the line, hence

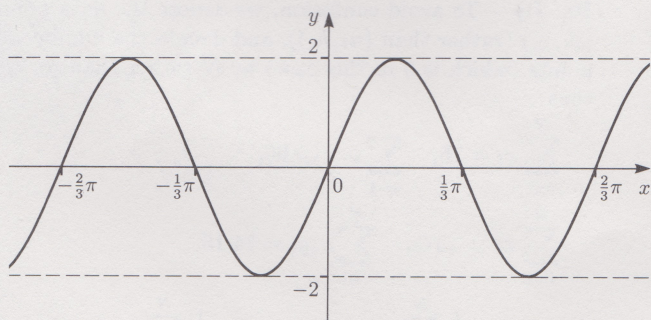
$$1 = -2 \times 1 + c,$$

i.e. $c = 3$.

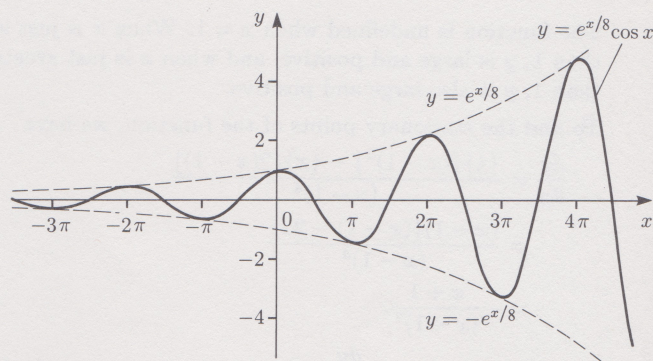
So the equation of the straight line is

$$y = -2x + 3.$$

3. (i) The curve $y = \sin x$ oscillates in the range $-1 \leq y \leq 1$ and crosses the x -axis at $x = 0, \pm\pi, \pm2\pi, \dots$. So the graph of $y = 2 \sin 3x$ oscillates in the range $-2 \leq y \leq 2$ and crosses the x -axis at $3x = 0, \pm\pi, \pm2\pi, \dots$, i.e. $x = 0, \pm\frac{1}{3}\pi, \pm\frac{2}{3}\pi, \dots$. Hence the graph is as sketched in Figure 3.

Figure 3 Sketch of the curve $y = 2 \sin 3x$.

(ii) The graph of $y = \cos x$ oscillates in the range $-1 \leq y \leq 1$ and crosses the x -axis at $x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \dots$. Hence the graph of $y = e^{x/8} \cos x$ oscillates in the range $-e^{x/8} \leq y \leq e^{x/8}$ and crosses the x -axis at $x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi$. So the sketch of the graph is as shown in Figure 4.

Figure 4 Sketch of the curve $y = e^{x/8} \cos x$.

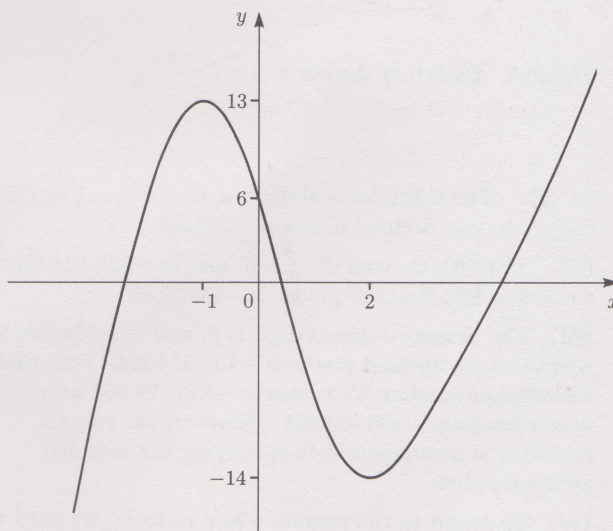
4. When $x = 0$, $y = 6$, and so the graph of $y = 2x^3 - 3x^2 - 12x + 6$ crosses the y -axis at $(0, 6)$. When $y = 0$, $2x^3 - 3x^2 - 12x + 6 = 0$. It is not easy to solve this cubic equation and so find where the graph crosses the x -axis.

For large x , we have $y \simeq 2x^3$. So when x is large and positive, y is also large and positive, and when x is large and negative, y is also large and negative.

The function is defined for all values of x .

Using the solution to Exercise 8 of Subsection 3.4, the function has a local maximum at $(-1, 13)$ and a local minimum at $(2, -14)$.

Collecting together the above information, we arrive at the sketch of the graph shown in Figure 5.

Figure 5 Sketch of the curve $y = 2x^3 - 3x^2 - 12x + 6$.

(Incidentally, we can conclude from the sketch that the graph must cross the x -axis at three points: one to the left of -1 , one between -1 and 2 , and one to the right of 2 .)

5. When $x = 0$, $y = 0$, and so the graph crosses the y -axis at the origin. When $y = 0$, $x = 0$, and so the curve crosses the x -axis only at the origin.

For large x ,

$$y \simeq \frac{x}{x^2} = \frac{1}{x} \simeq 0.$$

When x is large and positive, y is small and positive, whereas when x is large and negative, y is small and negative.

The function is undefined when $x = 1$. When x is just less than 1, y is large and positive, and when x is just greater than 1, y is also large and positive.

To find the stationary points of the function, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{\{1\}\{(x-1)^2\} - \{x\}\{2(x-1)\}}{(x-1)^4} \\ &= \frac{(x-1)\{(x-1) - 2x\}}{(x-1)^4} \\ &= -\frac{x+1}{(x-1)^3}.\end{aligned}$$

At stationary points, $\frac{dy}{dx} = 0$, which leads to $x = -1$. At $x = -1$, $y = -\frac{1}{4}$. For this function, it is probably easier to use the first derivative test to classify the stationary point, rather than the second derivative test. For x just less than -1 , $\frac{dy}{dx}$ is negative. For x just greater than -1 , $\frac{dy}{dx}$ is positive. So the function has a local minimum at $(-1, -\frac{1}{4})$. Collecting together all the above information, the sketch of the graph is as shown in Figure 6.

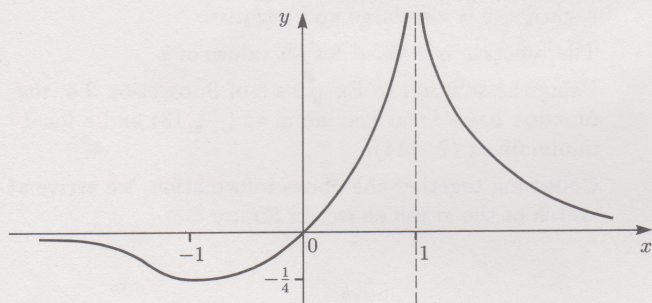


Figure 6 Sketch of the curve $y = \frac{x}{(x-1)^2}$.

6. (i) The third decimal digit is 4, and so -136.5248635 correct to two decimal places is -136.52 .

(ii) The fifth decimal digit is 6, and so -136.5248635 correct to four decimal places is -136.5249 .

(iii) The seventh decimal digit is 5, and so -136.5248635 correct to six decimal places is -136.524864 . Note that here the original number is as close to -136.524863 as it is to our approximation, -136.524864 . However, our rule for rounding is unambiguous in specifying the rounded approximation.

(iv) To round to the nearest whole number, we need to look at the first decimal digit, which is 5. So -136.5248635 is -137 to the nearest whole number.

7. My calculator gives the value of π as 3.141592654, which is correct to nine decimal places. So to four decimal places the value of π is 3.1416 and the error in this approximation is 0.000007 to six decimal places.

8. (i) To four significant figures, the approximation is 325 000 with an error of 39. Note that from the form 325 000 it is impossible to deduce that this approximation is correct to four significant figures, as the fourth digit is 0. If we wished to make this clear, we could rewrite it using scientific notation as 3.250×10^5 .

(ii) To four significant figures, the approximation is 0.001 743 with an error of $-0.000\,000\,498$.

(iii) To four significant figures, the approximation is -6.598×10^{-8} with an error of 3.74×10^{-12} .

9. (i)

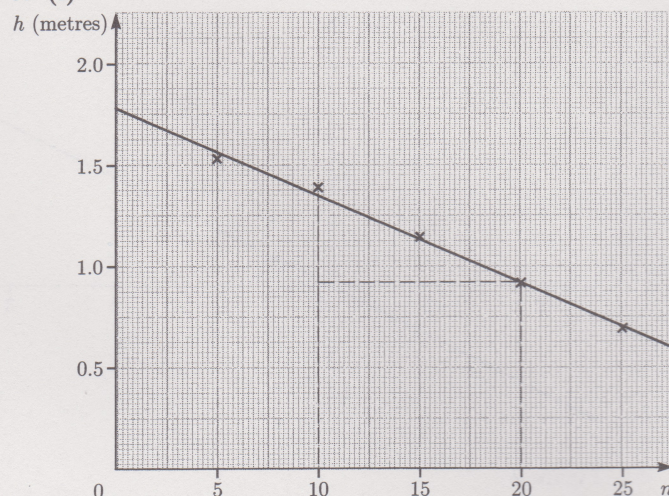


Figure 7

(ii) The plot of the height, h , of the bag against the number, n , of coins is shown by the crosses in Figure 7. The figure also includes my 'by eye' 'best' linear fit to the data.

(iii) The estimate of the height of the bag before it contains any coins is the y -intercept of the 'best' fit line, which in my case is

$$c = 1.78 \text{ (metres)}.$$

(iv) Two points on my line (corresponding to $n = 10$ and $n = 20$) are $(10, 1.35)$ and $(20, 0.92)$. So an increase of 10 coins results in a decrease of $1.35 - 0.92 = 0.43$ metres in the height of the bag above the floor. So the addition of one coin results in a decrease in the height of 0.043 metres.

(v) In part (iii) we found that the y -intercept of the line is $c = 1.78$, whereas in part (iv) we found that the slope of the line is $m = -0.043$. So my 'best' fit is

$$h = -0.043n + 1.78.$$

10. (i) To avoid confusion, we denote the data points by (x_i, y_i) (rather than (n_i, h_i)), and denote the number of data points (which is 5 in this case) by N rather than n . We find that

$$\begin{aligned}\sum_{i=1}^N x_i &= 75, & \sum_{i=1}^N y_i &= 5.66, \\ \sum_{i=1}^N x_i^2 &= 1375, & \sum_{i=1}^N x_i y_i &= 74.15.\end{aligned}$$

Hence $\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i = 15$ and $\bar{y} = \frac{1}{N} \sum_{i=1}^N y_i = 1.132$. So

$$m = \frac{\frac{1}{N} \sum_{i=1}^N x_i y_i - \bar{x} \bar{y}}{\frac{1}{N} \sum_{i=1}^N x_i^2 - \bar{x}^2} = -0.043,$$

$$\text{and } c = \bar{y} - m\bar{x} = 1.777.$$

So the best fit line is

$$h = -0.0430n + 1.78,$$

where we have rounded the final answers to three significant figures, which is all the experimental data justifies.

(ii) Using the best fit (with $c = 1.777$), we find that when $n = 12$ we have $h = 1.261$. So again rounding to three significant figures, the estimate for the height of the bag above the floor when it contains 12 coins is 1.26 metres.

11. $\frac{dy}{dx} = e^x.$

Hence the slope of the tangent to the curve at $x = 0$ is

$$m = \frac{dy}{dx}(0) = 1.$$

The equation of the tangent is therefore of the form

$$y = x + c.$$

In order to find the constant, we note that the tangent passes through the point $(0, y(0))$, i.e. $(0, 1)$. Hence

$$1 = 0 + c,$$

i.e. $c = 1$.

So the equation of the tangent to the curve $y = e^x$ at $x = 0$ is

$$y = x + 1.$$

12. (i) $f(x) = e^x \Rightarrow f(0) = 1$
 $f'(x) = e^x \Rightarrow f'(0) = 1$
 $f''(x) = e^x \Rightarrow f''(0) = 1$
 $f'''(x) = e^x \Rightarrow f'''(0) = 1$
 $f^{(4)}(x) = e^x \Rightarrow f^{(4)}(0) = 1$

(ii) $p_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \Rightarrow p_3(0) = a_0$
 $p'_3(x) = a_1 + 2a_2x + 3a_3x^2 \Rightarrow p'_3(0) = a_1$
 $p''_3(x) = 2a_2 + 6a_3x \Rightarrow p''_3(0) = 2a_2$
 $p'''_3(x) = 6a_3 \Rightarrow p'''_3(0) = 6a_3$

So the conditions $p_3(0) = f(0)$, $p'_3(0) = f'(0)$, $p''_3(0) = f''(0)$ and $p'''_3(0) = f'''(0)$ lead to

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2} \quad \text{and} \quad a_3 = \frac{1}{6}$$

respectively. So the cubic approximation is

$$p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

(iii) $p_4(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \Rightarrow p_4(0) = a_0$
 $p'_4(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \Rightarrow p'_4(0) = a_1$
 $p''_4(x) = 2a_2 + 6a_3x + 12a_4x^2 \Rightarrow p''_4(0) = 2a_2$
 $p'''_4(x) = 6a_3 + 24a_4x \Rightarrow p'''_4(0) = 6a_3$
 $p^{(4)}_4(x) = 24a_4 \Rightarrow p^{(4)}_4(0) = 24a_4$

Hence the conditions $p_4(0) = f(0)$, $p'_4(0) = f'(0)$, $p''_4(0) = f''(0)$, $p'''_4(0) = f'''(0)$ and $p^{(4)}_4(0) = f^{(4)}(0)$ lead to

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{6} \quad \text{and} \quad a_4 = \frac{1}{24}$$

respectively. Therefore the quartic approximation is

$$p_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.$$

In order to see the pattern to these polynomial approximations, it is better to write them in the form

$$p_3(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

and $p_4(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4.$

13. In order to write down the n th-order Taylor polynomial approximation to $f(x)$ about $x = 0$, we need to evaluate $f(0)$, $f'(0)$, $f''(0)$, $\dots$, $f^{(n-1)}(0)$ and $f^{(n)}(0)$.

$$\begin{aligned} f(x) &= \frac{1}{1-x} & \Rightarrow f(0) &= 1 \\ f'(x) &= \frac{1}{(1-x)^2} & \Rightarrow f'(0) &= 1 \\ f''(x) &= \frac{2}{(1-x)^3} & \Rightarrow f''(0) &= 2 \\ f'''(x) &= \frac{6}{(1-x)^4} = \frac{3!}{(1-x)^4} & \Rightarrow f'''(0) &= 3! \\ f^{(4)}(x) &= \frac{24}{(1-x)^5} = \frac{4!}{(1-x)^5} & \Rightarrow f^{(4)}(0) &= 4! \\ & & & \vdots \end{aligned}$$

$$f^{(n-1)}(x) = \frac{(n-1)!}{(1-x)^n} \Rightarrow f^{(n-1)}(0) = (n-1)!$$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \Rightarrow f^{(n)}(0) = n!$$

So the n th-order Taylor polynomial approximation about $x = 0$ to the function $f(x) = \frac{1}{1-x}$ is

$$\begin{aligned} p_n(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 1 + x + x^2 + x^3 + \dots + x^{n-1} + x^n. \end{aligned}$$

14. In order to find the Taylor series about $x = 0$ for the function $f(x) = \sin x$, we need to evaluate $f(0)$, and $f^{(n)}(0)$ for $n = 1, 2, 3, \dots$

$$\begin{aligned} f(x) &= \sin x & \Rightarrow f(0) &= 0 \\ f'(x) &= \cos x & \Rightarrow f'(0) &= 1 \\ f''(x) &= -\sin x & \Rightarrow f''(0) &= 0 \\ f'''(x) &= -\cos x & \Rightarrow f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & \Rightarrow f^{(4)}(0) &= 0 \end{aligned}$$

The above pattern of derivatives now repeats itself, and so we have

$$f^{(n)}(0) = \begin{cases} 0, & n = 2, 4, 6, \dots, \\ 1, & n = 1, 5, 9, \dots, \\ -1, & n = 3, 7, 11, \dots \end{cases}$$

So the Taylor series about $x = 0$ for the function $f(x) = \sin x$ is

$$\begin{aligned} f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \end{aligned}$$

15. (i) Replacing x by $-x$ in the Taylor series for e^x , we have

$$\begin{aligned} e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots \\ &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \end{aligned}$$

(ii) Replacing x by $-2x$ in the Taylor series for $\log_e(1+x)$, we obtain

$$\begin{aligned} \log_e(1-2x) &= (-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \dots \\ &= -2x - \frac{4}{2}x^2 - \frac{8}{3}x^3 - \frac{16}{4}x^4 - \dots \end{aligned}$$

We have not performed any cancellations in the coefficients as this would destroy their pattern.

(iii) Replacing x by x^2 in the Taylor series about $x = 0$ for $\sin x$ gives

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$

So

$$x \sin x^2 = x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + \dots$$

16. (i) To find the Taylor series about $x = 1$ for the function $f(x) = \log_e x$, we need to evaluate $f(1)$, and $f^{(n)}(1)$ for $n = 1, 2, 3, \dots$

$$\begin{aligned} f(x) = \log_e x &\Rightarrow f(1) = 0 \\ f'(x) = \frac{1}{x} &\Rightarrow f'(1) = 1 \\ f''(x) = -\frac{1}{x^2} &\Rightarrow f''(1) = -1 \\ f'''(x) = \frac{2}{x^3} &\Rightarrow f'''(1) = 2 \\ f^{(4)}(x) = -\frac{6}{x^4} = -\frac{3!}{x^4} &\Rightarrow f^{(4)}(1) = -3! \\ f^{(5)}(x) = \frac{24}{x^5} = \frac{4!}{x^5} &\Rightarrow f^{(5)}(1) = 4! \\ &\vdots \end{aligned}$$

Hence the Taylor series about $x = 1$ is

$$\begin{aligned} \log_e x &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &\quad + \frac{f'''(1)}{3!}(x-1)^3 + \dots \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots \end{aligned}$$

In fact, this Taylor series is valid only for $0 < x \leq 2$. There is no prior way that you could predict this, although you should have realized that it was not valid for $x \leq 0$, as $\log_e x$ is not defined in this range.

An alternative approach would be to replace x by $x-1$ in the (standard) Taylor series about $x = 0$ for $\log_e(1+x)$.

(ii) Putting $x = 1.1$ in the above Taylor series, and working to 6 decimal places, we have

$$\begin{aligned} \log_e(1.1) &= 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \dots \\ &= 0.1 - 0.005 + 0.000333 - 0.000025 + 0.000002 \\ &= 0.095310. \end{aligned}$$

So to 4 decimal places,

$$\log_e 1.1 = 0.0953.$$

In the above, we have worked throughout to 6 ($= 4 + 2$) decimal places, as our final answer was required to 4 decimal places. We ignored all terms in the Taylor series beyond $n = 5$, as these were smaller than this accuracy requires.

Solutions to the exercises in Section 6

1. The equation can be written in the form

$$x^4 - 2x^2 - 8 = 0,$$

which can be factorized to obtain

$$(x^2 - 4)(x^2 + 2) = 0.$$

Hence $x^2 = 4$ or $x^2 = -2$. As x^2 is non-negative, we can discount the second possibility. Therefore

$$x^2 = 4,$$

i.e. $x = 2$ or $x = -2$.

$$2. (i) {}^nC_r = \frac{n!}{r!(n-r)!}, \text{ so}$$

$${}^nC_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = {}^nC_r.$$

(ii) The above formula describes algebraically the fact that Pascal's triangle is symmetric about the vertical.

$$(iii) {}^{100}C_{98} = {}^{100}C_{(100-2)} = {}^{100}C_2 = \frac{100 \times 99}{2 \times 1} = 4950$$

3. We use the half-angle formula

$$\sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta.$$

In order to express this in terms of $t = \tan \frac{1}{2}\theta$, we recall that

$$\sec^2 \frac{1}{2}\theta = 1 + \tan^2 \frac{1}{2}\theta = 1 + t^2.$$

So

$$\begin{aligned} \sin \theta &= 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta \\ &= 2 \times \frac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} \times \cos^2 \frac{1}{2}\theta \\ &= 2 \times \tan \frac{1}{2}\theta \times \frac{1}{\sec^2 \frac{1}{2}\theta} \\ &= \frac{2t}{1+t^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \cos \theta &= 2 \cos^2 \frac{1}{2}\theta - 1 \\ &= \frac{2}{\sec^2 \frac{1}{2}\theta} - 1 \\ &= \frac{2}{1+t^2} - 1 \\ &= \frac{2 - (1+t^2)}{1+t^2} \\ &= \frac{1-t^2}{1+t^2}. \end{aligned}$$

Finally,

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \left(\frac{2t}{1+t^2} \right) \div \left(\frac{1-t^2}{1+t^2} \right) \\ &= \frac{2t}{1-t^2}. \end{aligned}$$

4. We can express the equation purely in terms of $\tan \theta$ by using the identity

$$\sec^2 \theta = 1 + \tan^2 \theta.$$

The equation becomes a quadratic equation in $\tan \theta$, namely

$$1 + \tan^2 \theta = 4 \tan \theta + 6$$

or $\tan^2 \theta - 4 \tan \theta - 5 = 0$.

So $(\tan \theta - 5)(\tan \theta + 1) = 0$,

i.e. $\tan \theta = -1$ or 5 .

Now $\tan \theta$ is positive in the first and third quadrants and negative in the second and fourth quadrants, so in the range $0 \leq \theta < 2\pi$, $\tan \theta = -1$ when $\theta = \frac{3}{4}\pi$ and $\frac{7}{4}\pi$, and $\tan \theta = 5$ when $\theta = \arctan 5$ ($\simeq 1.373$ radians $\simeq 78.69^\circ$) and $\pi + \arctan 5$ ($\simeq 4.515$ radians $\simeq 258.7^\circ$).

Hence the values of θ in the range $0 \leq \theta < 2\pi$ which satisfy the equation are

$$\arctan 5, \quad \frac{3}{4}\pi, \quad \pi + \arctan 5, \quad \frac{7}{4}\pi.$$

5. Suppose

$$2^x = e^z.$$

In order to express z in terms of x , we take logarithms to base e of both sides of this equation. Hence

$$\log_e(e^z) = \log_e(2^x),$$

$$\text{i.e. } z = x \log_e 2.$$

Hence

$$y = 2^x = e^{x \log_e 2},$$

and so

$$\frac{dy}{dx} = (\log_e 2) e^{x \log_e 2} = (\log_e 2) 2^x.$$

$$\begin{aligned} 6. \text{ (i)} \quad \frac{A}{x+1} + \frac{B}{x+2} &= \frac{A(x+2)}{(x+1)(x+2)} + \frac{B(x+1)}{(x+1)(x+2)} \\ &= \frac{A(x+2) + B(x+1)}{(x+1)(x+2)} \\ &= \frac{(A+B)x + (2A+B)}{(x+1)(x+2)} \end{aligned}$$

(ii) Using the result of part (i), if

$$\frac{A}{x+1} + \frac{B}{x+2} = \frac{1}{(x+1)(x+2)},$$

then

$$(A+B)x + (2A+B) = 1$$

for all values of x apart from -1 and -2 . Comparing the coefficients of x on both sides of this equation leads to

$$A+B=0,$$

whereas a comparison of the constant terms gives

$$2A+B=1.$$

Solving these simultaneous equations for A and B , we obtain

$$A=1 \quad \text{and} \quad B=-1.$$

Hence

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}.$$

(iii) Using the result of part (ii),

$$\begin{aligned} \int \frac{1}{(x+1)(x+2)} dx &= \int \frac{1}{x+1} dx - \int \frac{1}{x+2} dx \\ &= \log_e(x+1) - \log_e(x+2) + C \\ &= \log_e \frac{x+1}{x+2} + C. \end{aligned}$$

7. (i) The function is the 'function of a function'

$$y = \sin u,$$

where $u = x + x^2$.

$$\text{So } \frac{dy}{du} = \cos u = \cos(x + x^2) \quad \text{and} \quad \frac{du}{dx} = 1 + 2x.$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (1 + 2x) \cos(x + x^2).$$

(ii) The function is a sum. The second term of this sum is the composite function

$$z = u^2,$$

where $u = \sin x$.

$$\text{So } \frac{dz}{du} = 2u = 2 \sin x \quad \text{and} \quad \frac{du}{dx} = \cos x.$$

Hence

$$\frac{d}{dx}(\sin^2 x) = \frac{dz}{dx} = \frac{dz}{du} \frac{du}{dx} = 2 \sin x \cos x.$$

Therefore

$$\begin{aligned} \frac{d}{dx}(\sin x + \sin^2 x) &= \cos x + 2 \sin x \cos x \\ &= \cos x(1 + 2 \sin x). \end{aligned}$$

(iii) The function is the 'function of a function'

$$y = \sin u,$$

where $u = \sin x$.

$$\text{So } \frac{dy}{du} = \cos u = \cos(\sin x) \quad \text{and} \quad \frac{du}{dx} = \cos x.$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \cos x \cos(\sin x).$$

(iv) The function is the 'function of a function of a function'

$$y = \sin u,$$

where $u = \sin v$ and $v = \sin x$.

$$\text{So } \frac{dy}{du} = \cos u = \cos(\sin v) = \cos(\sin(\sin x)),$$

$$\frac{du}{dv} = \cos v = \cos(\sin x) \quad \text{and} \quad \frac{dv}{dx} = \cos x.$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx} = \cos x \cos(\sin x) \cos(\sin(\sin x)).$$

(v) The function is a quotient, so

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1)(1+x^2) - (1+x)(2x)}{(1+x^2)^2} \\ &= \frac{(1+x^2) - (2x+2x^2)}{(1+x^2)^2} \\ &= \frac{1-2x-x^2}{(1+x^2)^2}. \end{aligned}$$

(vi) The function is the triple product $y = uvw$, where

$$u = 1 + x^2, \quad v = e^{-x} \quad \text{and} \quad w = \sin 3x.$$

$$\text{So } \frac{du}{dx} = 2x, \quad \frac{dv}{dx} = -e^{-x} \quad \text{and} \quad \frac{dw}{dx} = 3 \cos 3x.$$

The formula for the derivative of a triple product is

$$\begin{aligned} \frac{dy}{dx} &= vw \frac{du}{dx} + uw \frac{dv}{dx} + uv \frac{dw}{dx} \\ &= (e^{-x})(\sin 3x)(2x) + (1+x^2)(\sin 3x)(-e^{-x}) \\ &\quad + (1+x^2)(e^{-x})(3 \cos 3x) \\ &= (-1+2x-x^2)e^{-x} \sin 3x + 3(1+x^2)e^{-x} \cos 3x. \end{aligned}$$

$$8. \text{ (i)} \quad \int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

(ii) In order to evaluate the integral we use the substitution

$$u = 1 + x.$$

$$\text{So } x = u - 1 \quad \text{and} \quad \frac{du}{dx} = 1, \text{ or } du = dx. \text{ Hence}$$

$$\begin{aligned} \int \frac{x}{1+x} dx &= \int \frac{u-1}{u} du \\ &= \int \left(1 - \frac{1}{u}\right) du \\ &= u - \log_e u + C \\ &= (1+x) - \log_e(1+x) + C \\ &= x - \log_e(1+x) + C', \end{aligned}$$

where $C' = C + 1$.

An alternative approach is to re-express the integrand as follows:

$$\begin{aligned}\int \frac{x}{1+x} dx &= \int \frac{(1+x)-1}{1+x} dx \\ &= \int \left(1 - \frac{1}{1+x}\right) dx \\ &= x - \log_e(1+x) + C.\end{aligned}$$

(iii) My first thought was to try the substitution $u = 1 + x^2$, but this did not seem to help! In my second attempt, I used the substitution

$$\begin{aligned}x &= \tan u. \\ \text{So } \frac{dx}{du} &= \sec^2 u, \text{ i.e. } dx = \sec^2 u du. \text{ Hence} \\ \int \frac{x^2}{1+x^2} dx &= \int \frac{\tan^2 u}{1+\tan^2 u} (\sec^2 u du) \\ &= \int \frac{\tan^2 u \sec^2 u}{\sec^2 u} du \\ &= \int \tan^2 u du \\ &= \int (\sec^2 u - 1) du \\ &= \tan u - u + C \\ &= x - \arctan x + C.\end{aligned}$$

Alternatively, we could rewrite the integrand in a similar way to the method used in part (ii):

$$\begin{aligned}\int \frac{x^2}{1+x^2} dx &= \int \frac{(1+x^2)-1}{1+x^2} dx \\ &= \int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= x - \arctan x + C.\end{aligned}$$

(iv) In order to evaluate the integral we use the substitution

$$\begin{aligned}u &= 1 + e^x. \\ \text{So } \frac{du}{dx} &= e^x, \text{ i.e. } du = e^x dx. \text{ Hence} \\ \int \frac{e^x}{1+e^x} dx &= \int \left(\frac{1}{1+e^x}\right) (e^x dx) \\ &= \int \frac{1}{u} du \\ &= \log_e u + C \\ &= \log_e(1+e^x) + C.\end{aligned}$$

(v) In order to evaluate the integral we use the substitution

$$\begin{aligned}u &= \log_e x. \\ \text{So } \frac{du}{dx} &= \frac{1}{x}, \text{ i.e. } du = \frac{1}{x} dx. \text{ Hence} \\ \int \frac{1}{x \log_e x} dx &= \int \left(\frac{1}{\log_e x}\right) \left(\frac{1}{x} dx\right) \\ &= \int \frac{1}{u} du \\ &= \log_e u + C \\ &= \log_e(\log_e x) + C.\end{aligned}$$

(vi) In order to evaluate the integral we first use the substitution

$$u = \log_e x.$$

$$\begin{aligned}\text{So } \frac{du}{dx} &= \frac{1}{x}, \text{ i.e. } du = \frac{1}{x} dx. \text{ Hence} \\ \int \frac{\log_e(\log_e x)}{x \log_e x} dx &= \int \frac{\log_e(\log_e x)}{\log_e x} \left(\frac{1}{x} dx\right) \\ &= \int \frac{\log_e u}{u} du.\end{aligned}$$

Although this integral is simpler, it is still not a standard integral, and so we use a second substitution

$$\begin{aligned}v &= \log_e u. \\ \text{So } \frac{dv}{du} &= \frac{1}{u}, \text{ i.e. } dv = \frac{1}{u} du. \text{ Hence} \\ \int \frac{\log_e u}{u} du &= \int (\log_e u) \left(\frac{1}{u} du\right) \\ &= \int v dv \\ &= \frac{1}{2} v^2 + C \\ &= \frac{1}{2} (\log_e u)^2 + C.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{\log_e(\log_e x)}{x \log_e(x)} dx &= \int \frac{\log_e u}{u} du \\ &= \frac{1}{2} (\log_e u)^2 + C \\ &= \frac{1}{2} (\log_e(\log_e x))^2 + C.\end{aligned}$$

(vii) In order to eliminate the square root, we use the substitution

$$u^2 = 1 + e^x$$

(rather than $u = 1 + e^x$). So, using implicit differentiation,

$$2u \frac{du}{dx} = e^x = u^2 - 1,$$

$$\text{hence } dx = \frac{2u}{u^2 - 1} du.$$

Therefore

$$\begin{aligned}\int \frac{1}{\sqrt{1+e^x}} dx &= \int \frac{1}{u} \left(\frac{2u}{u^2-1}\right) du \\ &= 2 \int \frac{1}{u^2-1} du.\end{aligned}$$

Now as $u > 1$, we can use the standard integral

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log_e \frac{x-a}{x+a} + C,$$

from Subsection 7.1 of the *Handbook*. So

$$\begin{aligned}\int \frac{1}{\sqrt{1+e^x}} dx &= \log_e \frac{u-1}{u+1} + C \\ &= \log_e \frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} + C.\end{aligned}$$

9. We have $x(t) = (A \cos 2t + B \sin t)e^t$, so

$$\dot{x}(t) = (-2A \sin 2t + B \cos t)e^t + (A \cos 2t + B \sin t)e^t.$$

The conditions $x(0) = 2$ and $\dot{x}(0) = 1$ lead to

$$A = 2 \quad \text{and} \quad A + B = 1,$$

so the constants are $A = 2$ and $B = -1$, i.e.

$$x(t) = (2 \cos 2t - \sin t)e^t.$$

$$10. (i) \quad \frac{1}{2} \log_e \frac{v^2 - 1}{v^2 + 1} = gt + c.$$

To satisfy the condition $v = 2$ when $t = 0$, we require

$$\frac{1}{2} \log_e \frac{4-1}{4+1} = 0 + c,$$

$$\text{so } c = \frac{1}{2} \log_e \frac{3}{5}.$$

(ii) So we have

$$\frac{1}{2} \log_e \frac{v^2 - 1}{v^2 + 1} = gt + \frac{1}{2} \log_e \frac{3}{5},$$

$$\text{i.e. } 2gt = \log_e \frac{v^2 - 1}{v^2 + 1} - \log_e \frac{3}{5} = \log_e \frac{5(v^2 - 1)}{3(v^2 + 1)}.$$

Taking exponentials of both sides gives

$$e^{2gt} = \exp \left(\log_e \frac{5(v^2 - 1)}{3(v^2 + 1)} \right) = \frac{5(v^2 - 1)}{3(v^2 + 1)},$$

$$\text{so } 5(v^2 - 1) = 3(v^2 + 1)e^{2gt}.$$

Collecting together all the terms involving v ,

$$(5 - 3e^{2gt})v^2 = 5 + 3e^{2gt}.$$

Hence

$$v^2 = \frac{5 + 3e^{2gt}}{5 - 3e^{2gt}},$$

$$\text{or } v = \sqrt{\frac{5 + 3e^{2gt}}{5 - 3e^{2gt}}},$$

where we have assumed that v is positive.

$$\begin{aligned} 11. \quad A \cos(\theta + \phi) &= A(\cos \theta \cos \phi - \sin \theta \sin \phi) \\ &= (A \cos \phi) \cos \theta + (-A \sin \phi) \sin \theta. \end{aligned}$$

For this to be equal to $\cos \theta + 2 \sin \theta$ for all values of θ , we require

$$A \cos \phi = 1 \quad \text{and} \quad -A \sin \phi = 2.$$

We need to solve these two equations for A and ϕ . To find A , we square and add the two equations to obtain

$$A^2(\cos^2 \phi + \sin^2 \phi) = 1^2 + 2^2,$$

$$\text{i.e. } A^2 = 5;$$

$$\text{so } A = \sqrt{5}.$$

Substituting this value of A into the two equations, we have

$$\cos \phi = \frac{1}{\sqrt{5}} \quad \text{and} \quad \sin \phi = -\frac{2}{\sqrt{5}}.$$

As $\cos \phi$ is positive and $\sin \phi$ is negative, the angle ϕ must be in the fourth quadrant. So

$$\phi = 2\pi - \arccos \left(\frac{1}{\sqrt{5}} \right) \quad (\simeq 5.176 \text{ radians} \simeq 296.57^\circ).$$

12. We first use the formula

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

to find an expression for v in terms of u and the constant f :

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u} = \frac{u - f}{uf},$$

$$\text{so } v = \frac{uf}{u - f}.$$

We can assume that the distance of the object from the lens is positive, i.e. $u > 0$. However, note that for some values of u , namely $u < f$, v is negative. This just means that the object and its image are on the same side of the lens (such an image is called a virtual image). Also, if $u = f$, the image is at infinity.

The distance between the object and its image is

$$\begin{aligned} y &= u + v \\ &= u + \frac{uf}{u - f} \\ &= \frac{u(u - f) + uf}{u - f} \\ &= \frac{u((u - f) + f)}{u - f} \\ &= \frac{u^2}{u - f}. \end{aligned}$$

In fact, this formula is valid only for $u > f$, as distances are taken to be positive. For $u < f$, the correct formula is

$$y = \frac{u^2}{f - u}.$$

Summarizing, the distance between object and image is

$$y = \begin{cases} \frac{u^2}{f - u}, & 0 < u < f, \\ \frac{u^2}{u - f}, & u > f. \end{cases}$$

Consider first the case $u > f$:

$$\begin{aligned} \frac{dy}{du} &= \frac{(2u)(u - f) - (u^2)(1)}{(u - f)^2} \\ &= \frac{u(2(u - f) - u)}{(u - f)^2} \\ &= \frac{u(u - 2f)}{(u - f)^2}. \end{aligned}$$

In the region $0 < u < f$, the derivative is the negative of this.

The derivative is equal to zero when $u = 0$ and $u = 2f$. We can discount the possibility $u = 0$ for physical reasons. So the only stationary point in the physical region is when $u = 2f$. To classify this stationary point, we use the first derivative test. When u is just less than $2f$, $\frac{dy}{du}$ is negative.

On the other hand, when u is just greater than $2f$, $\frac{dy}{du}$ is positive. So the stationary point $u = 2f$ is a *local* minimum. The corresponding distance between the object and image is

$$y = \frac{(2f)^2}{2f - f} = 4f.$$

However, before we conclude that this is the *overall* minimum, we must examine the values of y at the limits of its domain. When u is large and positive, y is also large and positive. When u is close to f (where the function $y(u)$ is undefined), y is large and positive. Finally, when u is small and positive, y is also small and positive.

So the overall minimum distance between object and image is effectively zero and occurs when the object is as close as possible to the lens.

Appendix 2: Solutions to the supplementary exercises

Solutions to the supplementary exercises in Section 1

1. Rearranging the equation so that only the term involving a is on one side of the equation, we have

$$\frac{1}{2}at^2 = s - ut,$$

$$\text{so } a = \frac{2(s - ut)}{t^2}.$$

$$\begin{aligned} 2. \quad 2u - 5v &= 19, & (1) \\ 3u + 4v &= -29. & (2) \end{aligned}$$

Multiplying Equation (1) by 3 and Equation (2) by 2 gives

$$\begin{aligned} 6u - 15v &= 57, & (3) \\ 6u + 8v &= -58. & (4) \end{aligned}$$

Subtracting Equation (3) from Equation (4) leads to

$$23v = -115,$$

$$\text{i.e. } v = -5.$$

Substituting this in Equation (1), we obtain

$$2u + 25 = 19,$$

$$\text{i.e. } 2u = -6 \text{ or } u = -3.$$

Hence the solution is $u = -3, v = -5$.

$$3. \quad (i) \quad x^2 + 6x + 5 = (x + 1)(x + 5)$$

$$(ii) \quad 2y^2 + 5y + 3 = (y + 1)(2y + 3)$$

$$(iii) \quad 3x^2 - x = x(3x - 1)$$

4. (i) Rearranging the equation gives

$$x^2 + 2x - 8 = 0,$$

$$\text{i.e. } (x + 4)(x - 2) = 0.$$

So $x = 2$ or $x = -4$.

$$(ii) \quad 6\lambda^3 + \lambda^2 - \lambda = \lambda(6\lambda^2 + \lambda - 1) = \lambda(3\lambda - 1)(2\lambda + 1),$$

so the equation is equivalent to

$$\lambda(3\lambda - 1)(2\lambda + 1) = 0.$$

$$\text{Hence } \lambda = 0, \lambda = \frac{1}{3} \text{ or } \lambda = -\frac{1}{2}.$$

5. (i) Putting $a = 1, b = x$ and using the appropriate row of Pascal's triangle gives

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

(ii) Here $a = 2, b = x$ and $n = 5$, so

$$\begin{aligned} (2 + x)^5 &= 2^5 + 5 \times 2^4 \times x + 10 \times 2^3 \times x^2 \\ &\quad + 10 \times 2^2 \times x^3 + 5 \times 2 \times x^4 + x^5 \\ &= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5. \end{aligned}$$

$$6. \quad (i) \quad 1.76251 \times 10^{-3}$$

$$(ii) \quad -2.75 \times 10^2$$

$$7. \quad (-5)^4 = 625 \text{ and } (-4)^5 = -1024.$$

So the greater number is $(-5)^4$ (although the number with greater *magnitude* is $(-4)^5$).

$$8. \quad (i) \quad \left(\frac{16}{x^{12}}\right)^{-1/4} = \left(\frac{x^{12}}{16}\right)^{1/4} = \frac{(x^{12})^{1/4}}{16^{1/4}} = \frac{x^{12/4}}{\sqrt[4]{16}} = \frac{x^3}{2}$$

$$\begin{aligned} (ii) \quad \frac{1}{3} \log_e 27 + \frac{1}{2} \log_e \frac{1}{9} &= \log_e (27^{1/3}) + \log_e \left(\frac{1}{9}\right)^{1/2} \\ &= \log_e \sqrt[3]{27} + \log_e \sqrt{\frac{1}{9}} \\ &= \log_e 3 + \log_e \frac{1}{3} \\ &= \log_e \left(3 \times \frac{1}{3}\right) \\ &= \log_e 1 = 0 \end{aligned}$$

$$\begin{aligned} (iii) \quad \exp \left\{ \frac{5}{2} \log_e (x^4) \right\} &= \exp \left\{ \log_e (x^4)^{5/2} \right\} \\ &= (x^4)^{5/2} = x^{4 \times \frac{5}{2}} = x^{10} \end{aligned}$$

$$\begin{aligned} 9. \quad 5^2 + 10^2 + 15^2 + \cdots + 95^2 + 100^2 \\ &= \sum_{r=1}^{20} (5r)^2 \\ &= 25 \sum_{r=1}^{20} r^2 \\ &= 25 \times \frac{1}{6} \times 20 \times (20 + 1) \times (2 \times 20 + 1) \\ &= 25 \times \frac{1}{6} \times 20 \times 21 \times 41 = 71\,750, \end{aligned}$$

where we have used the formula

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

with $n = 20$.

$$\begin{aligned} 10. \quad 0.9999 \dots &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots \\ &= 9 \left\{ \left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \left(\frac{1}{10}\right)^4 + \cdots \right\} \\ &= 9 \times \frac{\frac{1}{10}}{1 - \frac{1}{10}} = 9 \times \frac{\frac{1}{10}}{\frac{9}{10}} = 1, \end{aligned}$$

$$\text{i.e. } 0.9999 \dots = 1.$$

This confirms a surprising result which you should know already; it generalizes to any decimal expression ending in an infinite repeating sequence of 9s, so that (for example)

$$3.141\,599\,999 \dots = 3.1416.$$

Solutions to the supplementary exercises in Section 2

$$1. \quad (i) \quad \cot \frac{1}{3}\pi = \frac{1}{\tan \frac{1}{3}\pi} = \frac{1}{\sqrt{3}}$$

$$(ii) \quad \sin\left(-\frac{2}{3}\pi\right) = -\sin \frac{1}{3}\pi = -\frac{\sqrt{3}}{2}$$

$$2. \quad (i) \quad \sin \frac{5}{3}\pi = -\sin \frac{1}{3}\pi = -\frac{\sqrt{3}}{2}, \text{ so}$$

$$\arcsin\left(\sin \frac{5}{3}\pi\right) = \arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{1}{3}\pi.$$

$$(ii) \quad \arccos \frac{1}{2} = \frac{1}{3}\pi, \text{ so}$$

$$\sin\left(\arccos \frac{1}{2}\right) = \sin\left(\frac{1}{3}\pi\right) = \frac{\sqrt{3}}{2}.$$

$$\begin{aligned} 3. \quad A \cos(\omega t + \phi) &= A(\cos \omega t \cos \phi - \sin \omega t \sin \phi) \\ &= (A \cos \phi) \cos \omega t - (A \sin \phi) \sin \omega t. \end{aligned}$$

This is of the form $B \cos \omega t + C \sin \omega t$, with

$$B = A \cos \phi \quad \text{and} \quad C = -A \sin \phi.$$

$$4. \quad \text{Putting } \beta = 2\alpha \text{ in the formula for } \cos(\alpha + \beta), \text{ we obtain}$$

$$\cos 3\alpha = \cos \alpha \cos 2\alpha - \sin \alpha \sin 2\alpha.$$

Now $\cos 2\alpha = 2 \cos^2 \alpha - 1$ and $\sin 2\alpha = 2 \sin \alpha \cos \alpha$, so

$$\begin{aligned} \cos 3\alpha &= \cos \alpha (2 \cos^2 \alpha - 1) - \sin \alpha (2 \sin \alpha \cos \alpha) \\ &= 2 \cos^3 \alpha - \cos \alpha - 2 \sin^2 \alpha \cos \alpha \\ &= 2 \cos^3 \alpha - \cos \alpha - 2(1 - \cos^2 \alpha) \cos \alpha \\ &= 4 \cos^3 \alpha - 3 \cos \alpha. \end{aligned}$$

Solutions to the supplementary exercises in Section 3

$$1. \quad (i) \quad z' = -9x^{-4} = -\frac{9}{x^4}$$

$$(ii) \quad \frac{d}{dt}(2 + 3 \sin t) = 3 \cos t$$

$$(iii) \quad \dot{x}(t) = -4 \sin t + 3 \cos t, \text{ hence}$$

$$\dot{x}(\tfrac{1}{2}\pi) = -4 \sin(\tfrac{1}{2}\pi) + 3 \cos(\tfrac{1}{2}\pi) = -4.$$

$$2. \quad (i) \quad \frac{dx}{dt} = \frac{3}{3t + 4}$$

$$(ii) \quad \frac{dy}{dx} = \frac{\frac{1}{2}}{1 + (\frac{1}{2}x)^2} = \frac{2}{4 + x^2}$$

$$(iii) \quad \frac{dy}{dt} = 2t \exp(t^2)$$

$$(iv) \quad \frac{dx}{dt} = -t \sec^2(3 - \tfrac{1}{2}t^2)$$

3. (i) Using the product rule and the chain rule,

$$\begin{aligned} \frac{dx}{dt} &= (1)(e^{3t}) + (t)(3e^{3t}) \\ &= (3t + 1)e^{3t}. \end{aligned}$$

(ii) Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{\left(\frac{1}{x}\right)(x^2 + 1) - (\log_e x)(2x)}{(x^2 + 1)^2} \\ &= \frac{x^2 + 1 - 2x^2 \log_e x}{x(x^2 + 1)^2}. \end{aligned}$$

(iii) Using the product rule and the chain rule,

$$\begin{aligned} g'(t) &= (1)(\exp(t^2)) + (t)(2t \exp(t^2)) \\ &= (2t^2 + 1) \exp(t^2). \end{aligned}$$

4. (i) Here we have to use the chain rule twice. First,

$$\frac{d}{dx}(\cos(y^2)) = \frac{d}{dy}(\cos(y^2)) \frac{dy}{dx}.$$

Using the chain rule for the second time,

$$\frac{d}{dy}(\cos(y^2)) = (-\sin(y^2))(2y).$$

So, finally,

$$\frac{d}{dx}(\cos(y^2)) = -2y \sin(y^2) \frac{dy}{dx}.$$

$$(ii) \quad \frac{d}{dx}(\log_e(x + e^y)) = \frac{1}{x + e^y} \frac{d}{dx}(x + e^y)$$

$$= \frac{1 + e^y \frac{dy}{dx}}{x + e^y}$$

$$(iii) \quad \frac{d}{dx}(\tan(xy)) = \sec^2(xy) \frac{d}{dx}(xy)$$

$$= (y + x \frac{dy}{dx}) \sec^2(xy)$$

5. The function is $y = f(x)$, where

$$f(x) = 3x^4 + 8x^3 - 48x^2 + 200.$$

$$\text{So } f'(x) = 12x^3 + 24x^2 - 96x$$

$$\text{and } f''(x) = 36x^2 + 48x - 96.$$

At stationary points, $f'(x) = 0$:

$$12x^3 + 24x^2 - 96x = 0,$$

$$\text{i.e. } 12x(x^2 + 2x - 8) = 0$$

$$\text{or } 12x(x - 2)(x + 4) = 0.$$

$$\text{So } x = 0, x = 2 \text{ or } x = -4.$$

Now $f''(0) = -96 = \text{negative}$ and $f(0) = 200$, so the stationary point $(0, 200)$ is a local maximum.

Also, $f''(2) = 36 \times 2^2 + 48 \times 2 - 96 = \text{positive}$ and $f(2) = 3 \times 2^4 + 8 \times 2^3 - 48 \times 2^2 + 200 = 120$, so the stationary point $(2, 120)$ is a local minimum.

Further, $f''(-4) = 36 \times (-4)^2 + 48 \times (-4) - 96 = \text{positive}$ and $f(-4) = 3 \times (-4)^4 + 8 \times (-4)^3 - 48 \times (-4)^2 + 200 = -312$, so the stationary point $(-4, -312)$ is a local minimum.

Note that the stationary points alternate in nature with increasing x , as you would expect.

Solutions to the supplementary exercises in Section 4

1. (i) Using the standard integral

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C$$

with $a = \frac{1}{3}$ gives

$$\int \sec^2 \tfrac{1}{3}x \, dx = 3 \tan \tfrac{1}{3}x + C.$$

(ii) Using the standard integral

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

with $a = 2$ and the variable y replacing the variable x , we obtain

$$\int \frac{1}{y^2 + 4} \, dy = \tfrac{1}{2} \arctan \tfrac{1}{2}y + C.$$

(iii) The last table of integrals in Subsection 7.1 of the *Handbook* contains the result

$$\int \sin \alpha x \cos \beta x \, dx = -\frac{\cos(\alpha - \beta)x}{2(\alpha - \beta)} - \frac{\cos(\alpha + \beta)x}{2(\alpha + \beta)} + C.$$

So

$$\begin{aligned} \int_0^{\pi/2} \sin 2x \cos 3x \, dx &= \left[-\frac{\cos(-x)}{(-2)} - \frac{\cos 5x}{10} \right]_0^{\pi/2} \\ &= \left[\tfrac{1}{2} \cos x - \tfrac{1}{10} \cos 5x \right]_0^{\pi/2} \\ &= \left(\tfrac{1}{2} \cos \tfrac{1}{2}\pi - \tfrac{1}{10} \cos \tfrac{5}{2}\pi \right) \\ &\quad - \left(\tfrac{1}{2} \cos 0 - \tfrac{1}{10} \cos 0 \right) \\ &= (0 - 0) - \left(\tfrac{1}{2} - \tfrac{1}{10} \right) \\ &= -\tfrac{2}{5}. \end{aligned}$$

2. (i) My initial guess is $(3x + 1)^{10}$. But

$$\frac{d}{dx}((3x + 1)^{10}) = 30(3x + 1)^9,$$

so my adjusted guess is $F(x) = \frac{1}{30}(3x + 1)^{10}$. As

$$\frac{d}{dx}\left(\frac{1}{30}(3x + 1)^{10}\right) = (3x + 1)^9,$$

I conclude that

$$\int (3x + 1)^9 dx = \frac{1}{30}(3x + 1)^{10} + C.$$

(ii) Concentrating on the reciprocal $(2x^2 + 1)^{-1}$ and recalling the standard integral

$$\int \frac{1}{x} dx = \log_e x + C,$$

my first guess is $\log_e(2x^2 + 1)$. However,

$$\frac{d}{dx}(\log_e(2x^2 + 1)) = \frac{4x}{2x^2 + 1},$$

so my adjusted guess is $F(x) = \frac{1}{4}\log_e(2x^2 + 1)$. Now

$$\frac{d}{dx}\left(\frac{1}{4}\log_e(2x^2 + 1)\right) = \frac{x}{2x^2 + 1},$$

and so

$$\int \frac{x}{2x^2 + 1} dx = \frac{1}{4}\log_e(2x^2 + 1) + C.$$

3. (i) Let $u = x^3 + 1$, so that $\frac{du}{dx} = 3x^2$ and $du = 3x^2 dx$.

Then

$$\begin{aligned} \int x^2 \sqrt{x^3 + 1} dx &= \frac{1}{3} \int (x^3 + 1)^{1/2} (3x^2 dx) \\ &= \frac{1}{3} \int u^{1/2} du \\ &= \frac{2}{9} u^{3/2} + C \\ &= \frac{2}{9} (x^3 + 1)^{3/2} + C. \end{aligned}$$

(ii) Let $u = t^2$, so that $\frac{du}{dt} = 2t$ and $du = 2t dt$. Hence the integral becomes

$$\begin{aligned} \int t e^{t^2} dt &= \frac{1}{2} \int (e^{t^2})(2t dt) \\ &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C \\ &= \frac{1}{2} e^{t^2} + C. \end{aligned}$$

(iii) Let $u = \sin \theta$, so that $\frac{du}{d\theta} = \cos \theta$ and $du = \cos \theta d\theta$.

Therefore

$$\begin{aligned} \int \cos \theta \sin^3 \theta d\theta &= \int (\sin \theta)^3 (\cos \theta d\theta) \\ &= \int u^3 du \\ &= \frac{1}{4} u^4 + C \\ &= \frac{1}{4} \sin^4 \theta + C. \end{aligned}$$

(iv) Let $u = \theta^2$, so that $\frac{du}{d\theta} = 2\theta$ and $du = 2\theta d\theta$. Thus

$$\begin{aligned} \int \theta \cos(\theta^2) d\theta &= \frac{1}{2} \int \cos(\theta^2) (2\theta d\theta) \\ &= \frac{1}{2} \int \cos u du \\ &= \frac{1}{2} \sin u + C \\ &= \frac{1}{2} \sin(\theta^2) + C. \end{aligned}$$

(v) We use the substitution $u = 2x - 1$ or $x = \frac{1}{2}u + \frac{1}{2}$, hence $dx = \frac{1}{2} du$. Therefore

$$\begin{aligned} \int \frac{x}{\sqrt{2x-1}} dx &= \int \frac{\frac{1}{2}u + \frac{1}{2}}{u^{1/2}} \left(\frac{1}{2} du\right) \\ &= \int \left(\frac{1}{4}u^{1/2} + \frac{1}{4}u^{-1/2}\right) du \\ &= \frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2} + C \\ &= \frac{1}{6}(2x-1)^{3/2} + \frac{1}{2}(2x-1)^{1/2} + C. \end{aligned}$$

(vi) We use the trigonometric substitution $x = \sin u$, so that $\frac{dx}{du} = \cos u$, i.e. $dx = \cos u du$.

Now when $x = 0$, $u = 0$, and when $x = \frac{1}{2}$, $u = \frac{1}{6}\pi$. So

$$\begin{aligned} \int_{x=0}^{x=1/2} \sqrt{1-x^2} dx &= \int_{u=0}^{u=\pi/6} \sqrt{1-\sin^2 u} (\cos u du) \\ &= \int_0^{\pi/6} \cos^2 u du \\ &= \frac{1}{2} \int_0^{\pi/6} (1 + \cos 2u) du \\ &= \frac{1}{2} \left[u + \frac{1}{2} \sin 2u \right]_{u=0}^{u=\pi/6} \\ &= \frac{1}{2} \left(\frac{1}{6}\pi + \frac{1}{2} \sin \frac{1}{3}\pi \right) - \frac{1}{2} (0 + \frac{1}{2} \sin 0) \\ &= \frac{1}{12}\pi + \frac{\sqrt{3}}{8}. \end{aligned}$$

4. (i) Using the priority list given in the text, we choose

$$u = x \quad \text{and} \quad \frac{dv}{dx} = e^{-2x},$$

so $\frac{du}{dx} = 1$ and $v = -\frac{1}{2}e^{-2x}$.

Substituting in the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

gives

$$\begin{aligned} \int x e^{-2x} dx &= -\frac{1}{2} x e^{-2x} + \frac{1}{2} \int e^{-2x} dx \\ &= -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C \\ &= -\frac{1}{4} (1 + 2x) e^{-2x} + C. \end{aligned}$$

(ii) We choose

$$u = x \quad \text{and} \quad \frac{dv}{dx} = \cos x,$$

so $\frac{du}{dx} = 1$ and $v = \sin x$.

Substituting in the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

we obtain

$$\begin{aligned} \int x \cos x dx &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C. \end{aligned}$$

Solutions to the supplementary exercises in Section 5

1. As the slope of the line is $m = 3$, its equation has the form

$$y = 3x + c.$$

The line passes through the point $(-1, 2)$, and so

$$2 = -3 + c, \quad \text{i.e. } c = 5.$$

So the equation of the straight line is

$$y = 3x + 5,$$

and the y -intercept is $c = 5$.

To find the x -intercept, we put $y = 0$:

$$0 = 3x + 5, \quad \text{i.e. } x = -\frac{5}{3}.$$

Hence the x -intercept is $-\frac{5}{3}$.

2. When $x = 0$, $y = 200$, and so the graph crosses the y -axis at $(0, 200)$. When $y = 0$, $3x^4 + 8x^3 - 48x^2 + 200 = 0$. It is not easy to solve this quartic equation to find where the curve crosses the x -axis.

When x is very large, $y \simeq 3x^4$. So when x is large and positive, y is large and positive. Similarly, when x is large and negative, y is again large and positive.

The function is defined for all values of x .

Using the solution to Supplementary Exercise 5 of Subsection 3.4, the function has a local maximum at $(0, 200)$ and local minima at $(-4, -312)$ and $(2, 120)$.

Collecting together the above information, we arrive at the sketch of the curve shown in Figure 1.

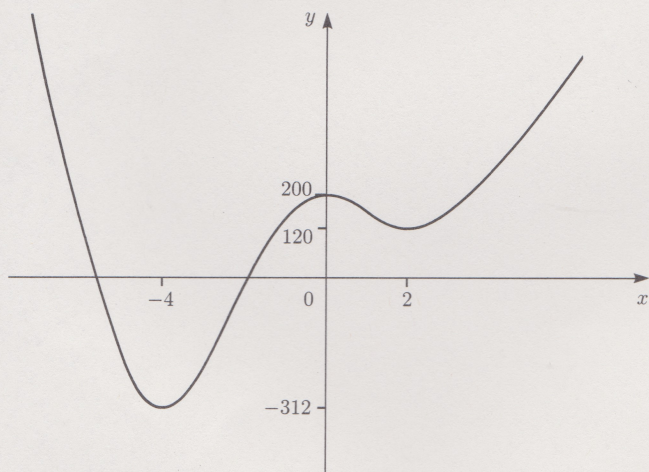


Figure 1 Sketch of the curve $y = 3x^4 + 8x^3 - 48x^2 + 200$.

3. When $x = 0$, $y = 0$, and so the graph crosses the y -axis at the origin. When $y = 0$, $x = 0$, and so the graph crosses the x -axis only at the origin.

When x is very large,

$$y \simeq \frac{x^2}{x} = x.$$

In particular, when x is large and positive, y is large and positive, and when x is large and negative, y is also large and negative.

The function is undefined for $x = -1$. When x is just less than -1 , y is large and negative. When x is just greater than -1 , y is large and positive.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(2x)(x+1) - (x^2)(1)}{(x+1)^2} \\ &= \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}, \end{aligned}$$

and at stationary points this must be equal to zero, which leads to $x = 0$ and $x = -2$. When $x = 0$, $y = 0$, and when $x = -2$, $y = -4$. So the stationary points are $(0, 0)$ and $(-2, -4)$. To classify these stationary points it is probably best to use the first derivative test.

When x is just less than 0, $\frac{dy}{dx}$ is negative, whereas when x is just greater than 0, $\frac{dy}{dx}$ is positive. Hence the stationary point $(0, 0)$ is a local minimum.

When x is just less than -2 , $\frac{dy}{dx}$ is positive, whereas when x is just greater than -2 , $\frac{dy}{dx}$ is negative. Hence the stationary point $(-2, -4)$ is a local maximum.

Collecting together the above information, we arrive at the sketch of the graph shown in Figure 2.

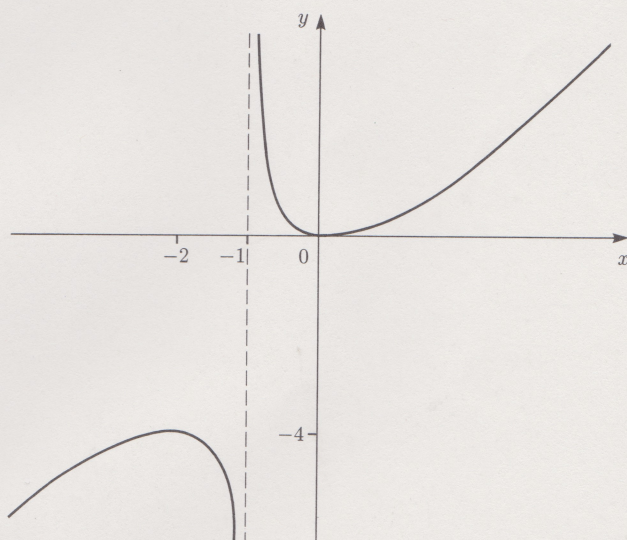


Figure 2 Sketch of the curve $y = \frac{x^2}{x+1}$.

$$\begin{aligned} 4. \quad f(x) &= \tan x & \Rightarrow f(0) &= 0 \\ f'(x) &= \sec^2 x & \Rightarrow f'(0) &= 1 \\ f''(x) &= (2 \sec x)(\sec x \tan x) & \Rightarrow f''(0) &= 0 \\ &= 2 \sec^2 x \tan x \\ f'''(x) &= 2\{(2 \sec^2 x \tan x)(\tan x) \\ &\quad + (\sec^2 x)(\sec^2 x)\} \\ &= 4 \sec^2 x \tan^2 x + 2 \sec^4 x & \Rightarrow f'''(0) &= 2 \end{aligned}$$

So the cubic Taylor approximation about $x = 0$ to the function $f(x) = \tan x$ is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x + \frac{1}{3}x^3.$$

5. Using the Taylor series for $(a+x)^r$ with $a=1$ and $r=\frac{1}{2}$, we have

$$\begin{aligned}(1+x)^{1/2} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 \\ &\quad + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}x^4 + \dots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots\end{aligned}$$

Putting $x=0.01$ and working throughout to 8 ($=6+2$) decimal places, we have

$$\begin{aligned}\sqrt{1.01} &= 1 + 0.005 - 0.0000125 + 0.00000006 \\ &= 1.00498756.\end{aligned}$$

So to 6 decimal places,

$$\sqrt{1.01} = 1.004988.$$

